Representing Poset Maps by Ring Homomorphisms

Jacob S. E. Saunders

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Introduction

Given a finite partially-ordered set (or poset), there are multiple methods to construct a commutative ring with a prime spectrum that is isomorphic to the poset. Lewis states in [1] that 'given two posets and an order preserving map between them, it is natural to ask whether we can find a ring homomorphism which induces the order preserving map'. In [2], Hochster gives an affirmative answer to this question, where it is shown that 'every spectral map of spectral spaces arises from a ring homomorphism' ([3], p69), and any order-preserving function between partially-ordered sets is induced by such a spectral map. Hochster's construction is '(in his own words) very intricate' ([3], p5), and so the rings involved are 'widely considered to be inaccessible'. In this paper we use a more straightforward method provided by Fontana in [4] known as the 'fibre product of rings' to construct rings isomorphic to a given poset. Then, given an order-preserving function, we observe the behaviour of the function on subsets of the poset. We use this to construct homomorphisms between subrings, combining them to form a homomorphism which corresponds to said order-preserving function.

In the first section, we provide an introduction to the relevant concepts from order theory and list some well-known results regarding prime spectra of rings and localisations of domains at prime ideals. We proceed to give a ring and homomorphism construction for the case in which we have a 1-dimensonal poset with a least element in the second section, showing that all order-preserving functions between such posets can be represented by a homomorphism. Then in the third section, we consider 1-dimensional posets which may have multiple minimal elements and show that, provided an order preserving function meets an additional requirement, we can construct a homomorphism which corresponds to it using similar methods as in the 'least element' case. In the fourth section we consider the n-dimensional case, and restrict ourselves to a class of posets which resemble 'upside-down trees'. Here we give a ring construction which provides us with a unique projection map for each element of the poset, making it straightforward to construct a homomorphism which induces a given order-preserving function, provided the order-preserving function belongs to a class of functions which we refer to as 'layer-compressing'.

1 Preliminary Material

We state some of the results in this introductory section without proof, given that proofs can be found in introductory commutative algebra textbooks (such as [5] or [6]).

1.1 Order Theory

We begin by looking at a concept which generalises equality called the 'equivalence relation'.

Definition 1.1 (Equivalence Relation)**.** Let $\sim$ be a binary relation on a set X. We call $\sim$ an equivalence relation if it satisfies the following properties:

1. (Reflexivity) if $x \in X$ then $x \sim x$;
2. (Symmetry) if $x, y \in X$ and $x \sim y$ then $y \sim x$;
3. (Transitivity) if $x, y, z \in X$ such that $x \sim y$ and $y \sim z$ then $x \sim z$.

On the other hand, a concept which generalises inequality is the 'partial order'.

Definition 1.2 (Partial Order and Poset)**.** Let $\leq$ be a binary relation on a set W. We call $\leq$ a partial order if it satisfies the following properties:

1. (Reflexivity) $w \in W$ then $w \leq w$;
2. (Anti-Symmetry) if $w_1, w_2 \in W$ with $w_1 \leq w_2$ and $w_2 \leq w_1$ then $w_1 = w_2$;
3. (Transitivity) if $w_1, w_2, w_3 \in W$ with $w_1 \leq w_2$ and $w_2 \leq w_3$ then $w_1 \leq w_3$.

We call $(W, \leq)$ a partially ordered set or poset. If the order relation is clear from context, then we may use W to refer to W equipped with $\leq$. Given a partial order $\leq$ we use $w_1 < w_2$ to denote $w_1 \leq w_2$ and $w_1 \neq w_2$. We refer to a subset of a poset with the same order relation as a subposet. All posets considered in this paper are of finite size.

It is worth noting that two elements of a set need not be related by a partial order - as opposed to a total order, which we define here.

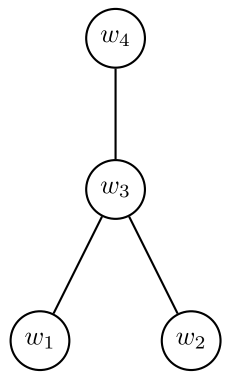
Definition 1.3 (Total Order)**.** Let $(W, \leq)$ be a poset. We call $\leq$ a total order on W if either $w_1 \leq w_2$ or $w_2 \leq w_1$ for all $w_1, w_2 \in W$, and we say W is totally ordered.

Definition 1.4. Let W be a poset and $w_1 \leq w_2$ be distinct elements of W. We say w_2 covers w_1 if there is no element of W strictly between w_1 and w_2, that is, if $w_1 \leq w_3$ and $w_1 \neq w_3$ then $w_2 \leq w_3$.

Throughout this paper we give visualisations of posets using Hasse diagrams. These are diagrams of a poset where the elements towards the top are the highest in the order, and the elements towards the bottom are the lowest in the order, where elements are connected by a line if one covers the other.

Example 1.5. Below is a Hasse diagram for the poset $W = \{w_1, w_2, w_3, w_4\}$ with the order relation

$$w_i \leq w_j \iff \begin{cases} i = 1 \text{ and } j = 1, 3, 4, \text{ or} \\ i = 2 \text{ and } j = 2, 3, 4, \text{ or} \\ i = 3 \text{ and } j = 3, 4, \text{ or} \\ i = 4 \text{ and } j = 4. \end{cases}$$



W

Definition 1.6 (Chain)**.** Given a poset W, a chain is a set of distinct elements $w_0, \ldots, w_n \in W$ such that $w_0 < w_2 < \cdots < w_n$. A chain of $n + 1$ elements has length n.

Definition 1.7 (Dimension of a Poset)**.** Let W be a poset. Then the dimension of W is the length of the longest chain in W.

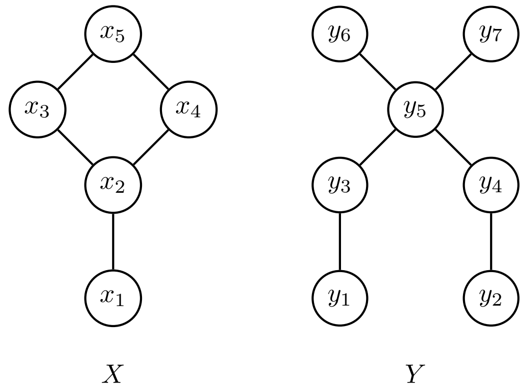
Definition 1.8 (Height of an Element)**.** Let W be a poset and $w \in W$. Then the height of w, denoted $\mathsf{ht}(w)$, is the length of the longest chain in W terminating at w.

Definition 1.9. Let W be a poset and let $w_1 \in W$. Then

- we call w_1 a minimal element of W if $w \leq w_1$ implies $w_1 = w$ for all $w \in W$;
- we call w_1 a maximal element of W if $w_1 \leq w$ implies $w_1 = w$ for all $w \in W$;
- we call w_1 the least element of W if $w_1 \leq w$ for all $w \in W$;
- we call w_1 the greatest element of W if $w \leq w_1$ for all $w \in W$.

A poset may or may not have a least/greatest element, but if such an element exists then it is unique.

Example 1.10. Below are two 3-dimensional posets. The poset X has least element x_1 and greatest element x_5. The poset Y has no greatest or least elements, but has minimal elements y_1, y_2 and maximal elements y_6, y_7.



Definition 1.11 (Disjoint Union of Posets)**.** Let $(X, \leq_X), (Y, \leq_Y)$ be posets. Then the disjoint union of X and Y, denoted $X \sqcup Y$, is the set $X \cup Y$ together with the partial order

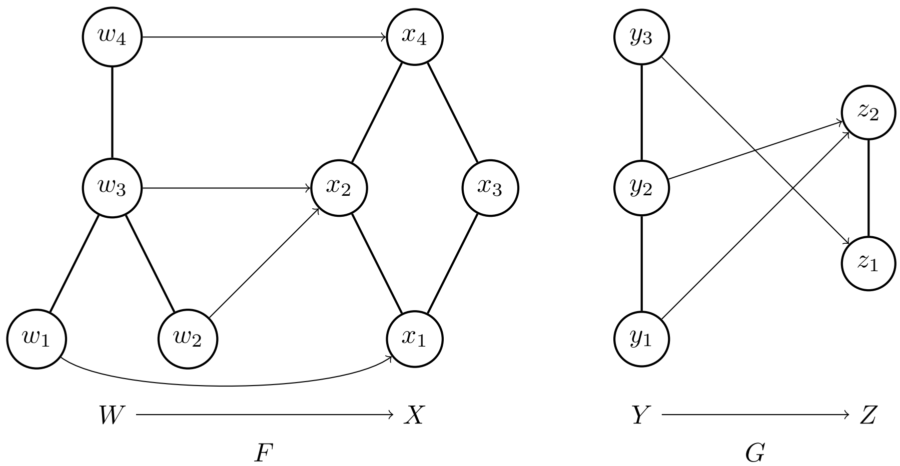
$$w_1 \leq_{X \sqcup Y} w_2 \iff \begin{cases} w_1, w_2 \in X \text{ and } w_1 \leq_X w_2, \text{ or} \\ w_1, w_2 \in Y \text{ and } w_1 \leq_Y w_2. \end{cases}$$

Definition 1.12 (Order-Preserving Function)**.** Let $(W, \leq), (\tilde{W}, \sqsubseteq)$ be posets and $F : W \to \tilde{W}$ a function between them. We say F is order-preserving if $w_1 \leq w_2$ implies $F(w_1) \sqsubseteq F(w_2)$.

We define an order-reflecting function similarly, where F is order-reflecting if $F(w_1) \sqsubseteq F(w_2)$ implies $w_1 \leq w_2$.

Definition 1.13 (Order Isomorphism)**.** Let $W, \tilde{W}$ be posets and $F : W \to \tilde{W}$ a function between them. If F is bijective, order-preserving and order-reflecting them we call F an order isomorphism. We call W and $\tilde{W}$ are isomorphic and denote this $W \cong \tilde{W}$.

Example 1.14. Below is a diagram of functions $F : W \to X$ and $G : Y \to Z$. The function F is order-preserving, but not order-reflecting as $F(w_1) = x_1 \leq x_2 = F(w_2)$. The function G is neither order-preserving, nor order-reflecting.



1.2 Prime Spectrum of a Ring

Basic results and terminology from commutative algebra are assumed to be known, such as the definition of a ring, (integral) domain, field, homomorphism, ideal and prime ideal, as well as the definition of a polynomial ring in n indeterminates. All rings in this paper are commutative. We let $\langle r_1, \ldots, r_n \rangle$ represent the ideal generated by the elements $r_1, \ldots, r_n$, and the ring which the ideal is contained in should be deducible from context. The ideal generated by all elements of the set H is denoted $\langle H \rangle$. We use $\mathcal{I}(R)$ to denote the set of ideals of the ring R.

Definition 1.15 (Prime Spectrum)**.** Let R be a ring. Then the prime spectrum of R, denoted $\mathsf{Spec}\, R$, is the set of prime ideals of R. We often consider $\mathsf{Spec}\, R$ together with the partial order $\subseteq$.

Lemma 1.16. *Let k be a field. Then* $\mathsf{Spec}\, k = \{\langle 0 \rangle\}$.

Proof. The zero ideal is prime in any field, and any non-zero ideal of a field contains a unit, so is not proper and is therefore not prime. □

We will frequently use several results about the ideal structure of product rings.

Lemma 1.17. *Let R, S be rings. Then K is an ideal of $R \times S$ if and only if $K = I \times J$ for some $I \in \mathcal{I}(R), J \in \mathcal{I}(S)$.*

Proof. Let K be an ideal of $R \times S$. We define the sets I and J as follows:

$$I = \{r \in R : (r, 0) \in K\},$$
$$J = \{s \in S : (0, s) \in K\}.$$

We first show that I and J are themselves ideals. Let $a \in I, r \in R$. Then $(a, 0) \in K$ and $(r, 0) \in R \times S$, so $(a, 0) \cdot (r, 0) = (ar, 0) \in K$. Thus $ar \in I$. Let $a, b \in I$. Then $(a, 0), (b, 0) \in K$, so $(a, 0) + (b, 0) = (a + b, 0) \in K$. Thus $a + b \in I$. Therefore I is an ideal of R. To show J is an ideal of S, repeat the above proof with the first and second elements swapped.

We now claim $K = I \times J$. Let $(r, s) \in I \times J$. Then $(r, 0), (0, s) \in K$, so $(r, 0) + (0, s) = (r, s) \in K$. Hence $I \times J \subseteq K$. Now let $(r, s) \in K$. Since $(1, 0) \in R \times S$, $(r, s) \cdot (1, 0) = (r, 0) \in K$, so $r \in I$. Similarly $s \in J$. Hence $(r, s) \in I \times J$ and $K \subseteq I \times J$. Therefore $K = I \times J$.

Suppose $I \in \mathcal{I}(R), J \in \mathcal{I}(S)$. Let $(a, b), (c, d) \in I \times J$. Then $(a, b) + (c, d) = (a + c, b + d)$. Since $a, c \in I$ we have $a + c \in I$, and similarly $b + d \in J$, so $(a + c, b + d) \in I \times J$. Let $(a, b) \in I \times J$ and $(r, s) \in R \times S$. Then $(a, b)(r, s) = (ar, bs)$. Since $ar \in I$ and $bs \in J$ we have $(ar, bs) \in I \times J$. □

Lemma 1.18. *Let R, S be rings and $K \in \mathcal{I}(R \times S)$. Then $K \in \mathsf{Spec}\, R \times S$ if and only if $K = I \times S$ or $I = R \times J$ for some $I \in \mathsf{Spec}\, R, J \in \mathsf{Spec}\, S$.*

Proof. Let $K \in \mathsf{Spec}\, R$. Then by Lemma 1.17, we have $K = I \times J$, where

$$I = \{r \in R : (r, 0) \in K\},$$
$$J = \{s \in S : (0, s) \in K\}.$$

Let $ab \in I$. Then $(ab, 0) = (a, 0)(b, 0) \in K$, so either $(a, 0)$ or $(b, 0) \in K$, thus either $a \in I$ or $b \in I$, so I is either a prime ideal or equal to the whole ring. The same can be said about J. Suppose both I and J are prime ideals. Both are proper, so there exist $a \notin I$ and $b \notin J$. Thus $(a, 0)(0, b) = (0, 0) \in I \times J$, but neither $(a, 0)$ or $(0, b)$ are elements of $I \times J$. Thus $I \times J$ is not a prime ideal, so exactly one of I and J must be equal to the entire ring.

Now suppose $I \in \mathsf{Spec}\, R$. Then let $K = I \times S$. Let $(a, b)(c, d) = (ac, bd) \in K$. Then $ac \in I$, so either $a \in I$ or $c \in I$. By virtue of (a, b) and (c, d) being elements of $R \times S$, we have $b, d \in S$. So either $(a, b) \in K$ or $(c, d) \in K$. Thus K is a prime ideal. The proof that $R \times J$ is a prime ideal when $J \in \mathsf{Spec}\, S$ is almost identical. □

Theorem 1.19. *Let R, S be rings. Then $\mathsf{Spec}\, R \times S \cong \mathsf{Spec}\, R \sqcup \mathsf{Spec}\, S$.*

Proof. The proof follows from the fact that the function

$$f : \mathsf{Spec}\, R \times S \to \mathsf{Spec}\, R \sqcup \mathsf{Spec}\, S,$$
$$f(I \times J) = \begin{cases} I & \text{if } J = S, \\ J & \text{if } I = R. \end{cases}$$

is an order isomorphism. □

Lemma 1.20. *Let R, S be rings and $\Phi : R \to S$ be a homomorphism. Then $\Phi^{-1}(I) \in \mathcal{I}(R)$ for all $I \in \mathcal{I}(S)$.*

Proof. Let $I \in \mathcal{I}(S)$. Let $r_1, r_2 \in \Phi^{-1}(I)$. Then there exist $s_1, s_2 \in I$ such that $\Phi(r_1) = s_1, \Phi(r_2) = s_2$. Then $s_1 + s_2 \in I$ and $\Phi(r_1 + r_2) = s_1 + s_2 \in I$, so $r_1 + r_2 \in \Phi^{-1}(I)$.
Now let $r_1 \in \Phi^{-1}(I)$ and $r_2 \in R$. Then there exists $s_1 \in I$ such that $\Phi(r_1) = s_1$. Define $s_2 = \Phi(r_2) \in S$. Then $\Phi(r_1 r_2) = s_1 s_2 \in I$, so $r_1 r_2 \in \Phi^{-1}(I)$. Therefore $\Phi^{-1}(I) \in \mathcal{I}(R)$. □

Theorem 1.21. *Let R, S be rings and $\Phi : R \to S$ be a homomorphism. Then $\Phi^{-1}(\mathfrak{p}) \in \mathsf{Spec}\, R$ for all $\mathfrak{p} \in \mathsf{Spec}\, S$.*

Proof. Let $\mathfrak{p} \in \mathsf{Spec}\, S$. If $1 \in \Phi^{-1}(\mathfrak{p})$ then $\Phi(1) = 1 \in \mathfrak{p}$, implying $\mathfrak{p}$ is not proper, which is a contradiction. Hence $\Phi^{-1}(\mathfrak{p})$ is proper. Let $r_1 r_2 \in \Phi^{-1}(\mathfrak{p})$. Then $\Phi(r_1)\Phi(r_2) = \Phi(r_1 r_2) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $\Phi(r_1) \in \mathfrak{p}$ or $\Phi(r_2) \in \mathfrak{p}$. Hence either $r_1 \in \Phi^{-1}(\mathfrak{p})$ or $r_2 \in \Phi^{-1}(\mathfrak{p})$, meaning $\Phi^{-1}(\mathfrak{p}) \in \mathsf{Spec}\, R$. □

This means any homomorphism induces a function between the spectra of its domain and codomain rings.

Definition 1.22 ('Spec' of a Homomorphism)**.** Let R, S be rings and $\Phi : R \to S$ be a homomorphism. Then we define the map $\mathsf{Spec}\, \Phi$ as follows:

$$\mathsf{Spec}\, \Phi : \mathsf{Spec}\, S \to \mathsf{Spec}\, R,$$
$$\mathsf{Spec}\, \Phi(\mathfrak{p}) = \Phi^{-1}(\mathfrak{p}).$$

It follows from a property of preimages that $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\Phi^{-1}(\mathfrak{p}) \subseteq \Phi^{-1}(\mathfrak{q})$, so $\mathsf{Spec}\,\Phi$ is always an order-preserving function.

1.3 Localisation of a Domain

Given a domain R, we can use a process called 'localisation' to extend the domain by introducing inverses of particular elements. Localising a domain modifies the prime spectrum in a predictable way, which will be useful for our ring constructions.

Definition 1.23 (Multiplicatively Closed Set)**.** Let R be a ring and $S \subseteq R$. We call S a multiplicatively closed subset of R if $1 \in S$ and $ab \in S$ for all $a, b \in S$.

Define the set

$$K = \left\{ \frac{r}{s} : r \in R, s \in S \right\}.$$

Definition 1.24 (Equivalence of Fractions)**.** Let $\frac{a}{b}, \frac{c}{d} \in K$. We say $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent fractions if $ad = bc$.

Lemma 1.25. *'Equivalence of fractions' is an equivalence relation on K.*

We define $S^{-1}R$ to be the set of equivalence classes of K under equivalence of fractions. We simply use $\frac{a}{b}$ to refer to the equivalence class of K containing $\frac{a}{b}$.

Theorem 1.26. *Let S be a multiplicatively closed subset of a domain R. Then $S^{-1}R$, together with the operations*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

forms a ring, known as the localisation of R at S.

Lemma 1.27. *Let R be a domain. Let $\mathfrak{p}_\lambda$ be a set of prime ideals of R with index set Λ. Then $S = R \setminus \bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ is a multiplicatively closed set.*

Proof. Let $S = R \setminus \bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$. Then $1 \in S$ as $1 \notin \mathfrak{p}_\lambda$ because $\mathfrak{p}_\lambda$ is a proper ideal. Let $a, b \in S$ and suppose $ab \notin S$. Then $ab \in \bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$, so $ab \in \mathfrak{p}_\lambda$ for some $\lambda \in \Lambda$. But $\mathfrak{p}_\lambda$ is a prime ideal, so either $a \in \mathfrak{p}_\lambda$ or $b \in \mathfrak{p}_\lambda$. Hence either $a \notin S$ or $b \notin S$, which is a contradiction. □

If $S = R \setminus \mathfrak{p}$ then we call $S^{-1}R$ the localisation of R at $\mathfrak{p}$, and denote $S^{-1}R$ by $R_\mathfrak{p}$ If $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and $S = R \setminus \bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ then we denote $S^{-1}R$ by $R_{\mathfrak{p}_1, \ldots, \mathfrak{p}_n}$.

The localisation procedure provides us with an inclusion homomorphism $i : R \to S^{-1}R$, which sends the element $r \in R$ to $\frac{r}{1} \in S^{-1}R$. It can be useful to consider R to be a subring of $S^{-1}R$ by identifying $r \in R$ with its image $\frac{r}{1}$ under i. Hence we can also take ideals of the ring R and consider them as subsets (but not necessarily ideals) of the ring $S^{-1}R$. Now we can discuss the ideals of these new rings.

Lemma 1.28. *Let J be a proper ideal of $S^{-1}R$. Then the set $I = J \cap R$ is an ideal of R disjoint from S such that $\langle I \rangle = J$, that is, the ideal generated by the image of I under the inclusion homomorphism i is equal to J.*

Proof. First we show I is an ideal of R. Let $a, b \in I$, $r \in R$. Then $a, b \in J$, so $a+b \in J$. Also $a+b \in R$, so $a+b \in J \cap R = I$ We also have $ar \in J$ and $ar \in R$, so $ar \in I$. Thus I is an ideal of R.
Next, we show I and S are disjoint. Now suppose there exists $a \in I \cap S$. Then $a \in I$, so $a \in J$. Then $1 = \frac{a}{a} \in J$, which is a contradiction as J was assumed to be proper.
Finally, we show $\langle I \rangle = J$. Let $\frac{a}{b} \in J$. Then $a = \frac{a}{b} b \in J$, so $a \in I$. Then $a \in \langle I \rangle$, so $\frac{a}{b} \in \langle I \rangle$. Now let $\frac{a}{b} \in \langle I \rangle$. Then we can represent $\frac{a}{b}$ as

$$\frac{a}{b} = \sum_{\lambda \in \Lambda} a_\lambda \frac{r_\lambda}{s_\lambda},$$

where $a_\lambda \in I, r_\lambda \in R$ and $s_\lambda \in S$ for all $\lambda \in \Lambda$. Then $a_\lambda \in I \subseteq J$, so $a_\lambda \frac{r_\lambda}{s_\lambda} \in J$ for all λ. Hence their sum, $\frac{a}{b}$, is an element of J. Therefore $\langle I \rangle = J$. □

Lemma 1.29. *Let $\mathfrak{p}$ be a prime ideal of $S^{-1}R$. Then $\mathfrak{q} = \mathfrak{p} \cap R$ is a prime ideal of R disjoint from S.*

Proof. By Lemma 1.28, $\mathfrak{q}$ is an ideal of R. If $s \in \mathfrak{q} \cap S$ then $s \in \mathfrak{p}$, so $1 = \frac{s}{s} \in \mathfrak{p}$, which is a contradiction as $\mathfrak{p}$ is proper. Hence $1 \notin \mathfrak{q}$ as $1 \in S$, so $\mathfrak{q}$ is also proper. Let $ab \in \mathfrak{q}$. Then $ab \in \mathfrak{p}$ so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Then since $a, b \in R$, we have $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Therefore $\mathfrak{q}$ is a prime ideal. □

Lemma 1.30. *Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R disjoint from S. Then $\langle \mathfrak{p} \rangle = \langle \mathfrak{q} \rangle$ implies $\mathfrak{p} = \mathfrak{q}$.*

Proof. Suppose $\mathfrak{p} \neq \mathfrak{q}$. We can assume without loss of generality that there exists $a \in \mathfrak{p} \setminus \mathfrak{q}$. We have $a \in \langle \mathfrak{p} \rangle = \langle \mathfrak{q} \rangle$, so we can write

$$a = \sum_{\lambda \in \Lambda} a_\lambda \frac{r_\lambda}{s_\lambda},$$

where $a_\lambda \in \mathfrak{q}, r_\lambda \in R$ and $s_\lambda \in S$. Now define

$$t_\lambda = \prod_{\mu \neq \lambda} s_\mu,$$
$$t = \prod_{\lambda \in \Lambda} s_\mu,$$

and we have $s_\lambda = \frac{t}{t_\lambda}$. Then we can write

$$a = \frac{1}{t} \sum_{\lambda \in \Lambda} a_\lambda r_\lambda t_\lambda,$$

and we have $at \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime, either $a \in \mathfrak{q}$ or $t \in \mathfrak{q}$. But $t \in S$, so we must have $a \in \mathfrak{q}$, which is a contradiction. □

Lemma 1.31. *If $\mathfrak{p}$ is a prime ideal of R disjoint from S, then $\mathfrak{p} = \langle \mathfrak{p} \rangle \cap R$.*

Proof. Let $a \in \mathfrak{p}$. Then $a \in R$ and $a \in \langle \mathfrak{p} \rangle$, so $a \in \langle \mathfrak{p} \rangle \cap R$. Now let $a \in \langle \mathfrak{p} \rangle \cap R$. Then by similar arguments to the previous proof, we have $at \in \mathfrak{p}$ for some $t \in S$. Then since $\mathfrak{p}$ is prime, either $a \in \mathfrak{p}$ or $t \in \mathfrak{p}$. Since $\mathfrak{p}$ is disjoint from S, we have $a \in \mathfrak{p}$, so $\mathfrak{p} = \langle \mathfrak{p} \rangle \cap R$. □

Lemma 1.32. *If the ideal $\mathfrak{p}$ of R is prime and disjoint from S then $\langle \mathfrak{p} \rangle$ is prime.*

Proof. Let $\mathfrak{p}$ be prime and disjoint from S. Suppose $1 \in \langle \mathfrak{p} \rangle$. Then by similar arguments to the previous proof, there exists $1 \cdot t \in \mathfrak{p}$, which is a contradiction as we assumed $\mathfrak{p}$ to be disjoint from S. Now let $\frac{a}{b}\frac{c}{d} \in \langle \mathfrak{p} \rangle$. Then $ac \in \langle \mathfrak{p} \rangle$, and by similar arguments to the previous proof, we have that $act \in \mathfrak{p}$ for $t \in S$. Then either $ac \in \mathfrak{p}$ or $t \in \mathfrak{p}$, but $t \in \mathfrak{p}$ contradicts an assumption, so $ac \in \mathfrak{p}$. Then either $a \in \mathfrak{p}$ or $c \in \mathfrak{p}$, so either $\frac{a}{b} \in \langle \mathfrak{p} \rangle$ or $\frac{c}{d} \in \langle \mathfrak{p} \rangle$. Hence $\langle \mathfrak{p} \rangle$ is prime. □

Proposition 1.33. *There is a bijection between the prime ideals of $S^{-1}R$ and the prime ideals of R which are disjoint from S.*

Proof. Let $\mathcal{K} = \{\mathfrak{p} \in \mathsf{Spec}\, R : \mathfrak{p} \cap S = \emptyset\}$. The function

$$f : \mathcal{K} \to \mathsf{Spec}\, S^{-1}R,$$
$$f(\mathfrak{p}) = \langle \mathfrak{p} \rangle$$

is well-defined by Lemma 1.32, surjective by Lemma 1.28 together with Lemma 1.29, and injective by Lemma 1.29. □

Lemma 1.34. *Let $\mathfrak{p}, \mathfrak{q} \in \mathsf{Spec}\, R$ be disjoint from S. Then $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $\langle \mathfrak{p} \rangle \subseteq \langle \mathfrak{q} \rangle$.*

Proof. Assume $\mathfrak{p} \subseteq \mathfrak{q}$. Let $\frac{a}{b} \in \langle \mathfrak{p} \rangle$. Then $a \in \langle \mathfrak{p} \rangle$, and since $a \in R$ we have $a \in \langle \mathfrak{p} \rangle \cap R = \mathfrak{p} \subseteq \mathfrak{q}$. Then $\frac{a}{b} \in \langle \mathfrak{q} \rangle$. Now assume $\langle \mathfrak{p} \rangle \subseteq \langle \mathfrak{q} \rangle$. Let $a \in \mathfrak{p}$. Then $a \in \langle \mathfrak{p} \rangle \subseteq \langle \mathfrak{q} \rangle$. Since $a \in \langle \mathfrak{q} \rangle$ and $a \in R$, we have $a \in \langle \mathfrak{q} \rangle \cap R = \mathfrak{q}$. □

2 1-Dimensional Posets with a Least Element

This ring construction is remarkably simple, as the ability to localise domains quickly provides us with a ring having spectrum isomorphic to a given 1-dimensional poset with a least element. We show later that any order-preserving function between such posets is induced by an evaluation homomorphism.

Before beginning the ring construction, we show that all 1-dimensional posets with a least element can be categorised based on the number of elements they contain.

Lemma 2.1. *All finite 1-dimensional posets $(X, \leq)$ of size $n+1$ with a least element are of the form*

$$X = \{x_0, \ldots, x_n\},$$
$$x_i \leq x_j \iff i = 0 \text{ or } i = j.$$

Proof. Let Y be a 1-dimensional poset of size $n+1$ with a least element. Let y_0 be the least element of Y and denote the remaining elements $y_1, \ldots, y_n$ in any arbitrary manner. We claim that the function

$$\omega : X \to Y,$$
$$\omega(x_i) = y_i,$$

is an order isomorphism. It is bijective as it is a surjection between two sets of the same size, but it remains to be shown that it preserves and reflects order. Assume $x_i \leq x_j$. Then $i = 0$ or $i = j$. If $i = 0$ then $\omega(x_i) = y_0$ and $\omega(x_j) = y_j$, and as y_0 is the least element of Y, we have $y_0 \leq y_j$. If $i = j$ then $\omega(x_i) = \omega(x_j)$, so $\omega(x_i) \leq \omega(x_j)$. Therefore ω is order preserving.
Now suppose $\omega(x_i) \leq \omega(x_j)$. If $\omega(x_i) = y_0$ then $x_i = x_0 \leq x_j$. If $\omega(x_i) \neq y_0$ and $i \neq j$ then there exists a chain $y_0 < \omega(x_i) < \omega(x_j)$ of length 2, which is a contradiction as we assumed Y was 1-dimensional. Thus $i = j$ meaning $x_i \leq x_j$. Thus ω is order-reflecting, and is therefore an order isomorphism. □

2.1 Ring Construction

When we refer to a polynomial ring over k, then k is some arbitrary field.

Lemma 2.2. *The ideal $\langle X_i \rangle$ is prime in $k[X_1, \ldots, X_n]$.*

Proof. The proof follows from the fact that

$$\phi : k[X_1, \ldots, X_n] \to k[Y_1, \ldots, Y_{n-1}],$$
$$\phi(f(X_1, \ldots, X_n)) = f(Y_1, \ldots, Y_{i-1}, 0, Y_i, \ldots, Y_{n-1}),$$

is a surjective homomorphism with kernel $\langle X_i \rangle$, so

$$k[X_1, \ldots, X_n] / \langle X_i \rangle \cong k[Y_1, \ldots, Y_{n-1}].$$

Because $k[Y_1, \ldots, Y_{n-1}]$ is a domain, $\langle X_i \rangle$ is prime. □

Lemma 2.3. *All ideals $\langle X_i \rangle$ have height 1 in* $\mathsf{Spec}\, k[X_1, \dots, X_n]$.

Proof. Note that $k[X_1, \dots, X_n]$ is a domain, so $\langle 0 \rangle \in \mathsf{Spec}\, k[X_1, \dots, X_n]$. We have $\langle 0 \rangle \subset \langle X_i \rangle$, so $\mathsf{ht}\, \langle X_i \rangle \geq 1$. If $\mathsf{ht}\, \langle X_i \rangle > 1$ then there exists $\mathfrak{p} \in k[X_1, \dots, X_n]$ such that $\langle 0 \rangle \subset \mathfrak{p} \subset \langle X_i \rangle$. Let $f \in \mathfrak{p} \subseteq \langle X_1 \rangle$ be non-zero and suppose f is of degree n. Then $f = X_i f_1$ for some $f_1 \in k[X_1, \dots, X_n]$. Then as $\mathfrak{p}$ is prime, either $f_1 \in \mathfrak{p}$ or $X_i \in \mathfrak{p}$. If $X_i \in \mathfrak{p}$ then $\langle X_i \rangle = \mathfrak{p}$, so assume $f_1 \in \mathfrak{p}$. We can repeat this argument to find $f_n \in k[X_1, \dots, X_n]$ such that $f = X_i^{n+1} f_{n+1}$, which is a contradiction as f is of degree n. Therefore no such ideal exists, so $\mathsf{ht}\, \langle X_i \rangle = 1$. □

Theorem 2.4. $\mathsf{Spec}\, k[X_1, \dots, X_n]_{\langle X_1 \rangle, \dots, \langle X_n \rangle} = \{\langle 0 \rangle, \langle X_1 \rangle, \dots, \langle X_n \rangle\}$.

Proof. The zero ideal is prime as $k[X_1, \dots, X_n]$ is a domain. By Lemma 2.2 $\langle X_i \rangle$ are prime ideals of $k[X_1, \dots, X_n]$. Let $S = R \setminus \bigcup_{i=1}^{n} \langle X_i \rangle$ and let $R = S^{-1} k[X_1, \dots, X_n]$. Then $\langle 0 \rangle$ and $\langle X_i \rangle$ are disjoint from S, so by Lemma 1.32, all specified ideals are prime in R. It remains to be shown that there are no more prime ideals of R. We can do this by showing that there are no more prime ideals of $k[X_1, \dots, X_n]$ disjoint from S. Suppose $\mathfrak{p}$ is such an ideal. Then $\mathfrak{p} \cap S = \emptyset$, so $\mathfrak{p} \subseteq \bigcup_{i=1}^{n} \langle X_i \rangle$. Then by the prime avoidance theorem ([5] 3.61), $\mathfrak{p} \subseteq \langle X_i \rangle$ for some i. Then, since $\langle X_i \rangle$ has height 1 in $\mathsf{Spec}\, k[X_1, \dots, X_n]$, we must have $\mathfrak{p} = \langle 0 \rangle$ or $\mathfrak{p} = \langle X_i \rangle$. □

Proposition 2.5. *Let W be a 1-dimensional poset with $n+1$ elements and a least element. Then* $W \cong \mathsf{Spec}\, k[X_1, \dots, X_n]_{\langle X_1 \rangle, \dots, \langle X_n \rangle}$.

Proof. By Theorem 2.4, $\mathsf{Spec}\, k[X_1, \dots, X_n]_{\langle X_1 \rangle, \dots, \langle X_n \rangle}$ is a poset with $n+1$ elements. Also Lemma 2.3 tells us that $\mathsf{Spec}\, k[X_1, \dots, X_n]_{\langle X_1 \rangle, \dots, \langle X_n \rangle}$ is 1-dimensional and has least element $\langle 0 \rangle$, so the proof follows directly from Lemma 2.1. □

Theorem 2.6. *Let W be a 1-dimensional poset with $n+1$ elements and a least element and let $m \in \mathbb{N}_0$. Then* $W \cong \mathsf{Spec}\, k[X_1, \dots, X_{n+m}]_{\langle X_1 \rangle, \dots, \langle X_n \rangle}$.

Proof. Let

$$R = k[X_1, \dots, X_{n+m}],$$
$$S = R \setminus \bigcup_{i=1}^{n} \langle X_i \rangle,$$
$$T = R \setminus \bigcup_{i=1}^{n+m} \langle X_i \rangle.$$

Suppose $\langle \mathfrak{p} \rangle \in \mathsf{Spec}\, S^{-1}R$. Then $\mathfrak{p} \cap S = \emptyset$, and as $T \subseteq S$, we also have $\mathfrak{p} \cap T = \emptyset$, so $\langle \mathfrak{p} \rangle$ is a prime ideal of $T^{-1}R$ by Lemma 1.32. We know from Theorem 2.4 that the prime ideals of $T^{-1}R$ are $\langle 0 \rangle, \langle X_1 \rangle, \dots, \langle X_{n+m} \rangle$, so we need only check these. The ideals $\langle 0 \rangle, \langle X_1 \rangle, \dots, \langle X_n \rangle$ are prime in R and disjoint from S, so by Lemma 1.32 they are prime in $S^{-1}R$, but the ideals

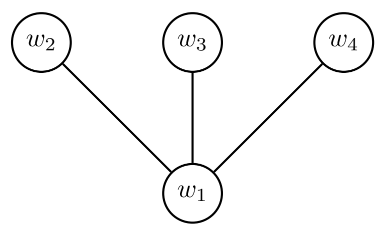
$\langle X_{n+1}\rangle, \dots, \langle X_{n+m}\rangle$ are not disjoint from S, so cannot be prime in $S^{-1}R$. Since $\mathsf{Spec}\, k[X_1, \dots, X_{n+m}]_{\langle X_1\rangle,\dots,\langle X_n\rangle} \cong \mathsf{Spec}\, k[X_1, \dots, X_n]_{\langle X_1\rangle,\dots,\langle X_n\rangle}$, the proof follows directly from Proposition 2.5. □

In other words, this gives us the ability to add 'throw-away' units to our rings whilst preserving the structure of the prime spectrum. This will come in handy when considering homomorphisms, allowing us to divert elements of the input ring to a place where they will not interfere with the preimages of prime ideals.

Example 2.7. The poset W as shown below has is 1-dimensional, has 4 elements and least element w_1, so

$$W \cong \mathsf{Spec}\, k[X_1, X_2, X_3]_{\langle X_1\rangle,\langle X_2\rangle,\langle X_3\rangle}.$$

In fact $W \cong \mathsf{Spec}\, k[X_1, \dots, X_m]_{\langle X_1\rangle,\langle X_2\rangle,\langle X_3\rangle}$ for all $m \geq 3$.



W

2.2 Homomorphism Construction

To know if our construction is valid, we need a formal way of verifying if $\mathsf{Spec}\,\Phi$ behaves in the same way as a given order-preserving function.

Definition 2.8 (Isomorphism of Functions)**.** Let W, X, Y, Z be posets and $F : W \to X, G : Y \to Z$ order-preserving functions. If there exist order isomorphisms $\omega : W \to Y$ and $\chi : X \to Z$ such that

$$F(w) = \chi^{-1} \circ G \circ \omega(w)$$

for all $w \in W$, then we say F is isomorphic to G.

Isomorphism of functions defines an equivalence relation on the set of order-preserving functions from a poset isomorphic to W to a poset isomorphic to X.

The homomorphism we intend to construct essentially corresponds to an evaluation homomorphism, which we define here.

Definition 2.9 (Evaluation Homomorphism[1])**.** Let $R \subseteq S$ be rings and let $\tilde{n} \in \mathbb{N}$. Then given any collection $s_1, \ldots, s_{\tilde{n}} \in S$, the unique homomorphism

$$\Psi : R[X_1, \ldots, X_{\tilde{n}}] \to S,$$
$$\Psi(r) = r \text{ for all } r \in R,$$
$$\Psi(f(X_1, \ldots, X_{\tilde{n}})) = f(s_1, \ldots, s_{\tilde{n}}),$$

is known as the evaluation homomorphism (or simply evaluation) at $s_1, \ldots, s_{\tilde{n}}$.

Our aim is to show that our evaluation homomorphism $\Psi : k[X_1, \ldots, X_{\tilde{n}}] \to D$ can be extended to a homomorphism $\Phi : \tilde{D} \to D$. To do this we use a property commonly known as the 'universal property of localisation'.

Proposition 2.10 (Universal Property of Localisation[2])**.** *Let S be a multiplicatively closed subset of a ring R; also let $i : R \to S^{-1}R$ denote the inclusion homomorphism $\left(i(r) = \frac{r}{1}\right)$. Let R' be a second ring, and let $\Psi : R \to R'$ be a homomorphism with the property that $\Psi(s)$ is a unit of R' for all $s \in S$. Then there is a unique homomorphism $\Phi : S^{-1}R \to R'$ such that $\Phi \circ f = \Psi$. In fact, Φ is such that*

$$\Phi\left(\frac{r}{s}\right) = \frac{\Psi(r)}{\Psi(s)}.$$

To make it easier to show that an evaulation Ψ sends elements of S to units of D, we give some results about 'algebraic independence'.

Definition 2.11 (Algebraic Independence[3])**.** Let $S = \{s_i\}_{i=1}^{n}$ be a family of elements of a ring R. Let R_0 be a subring of R. Then S is algebraically independent over R_0 if, given $f(X_1, \ldots, X_n) \in R_0[X_1, \ldots, X_n]$, the property

$$f(s_1, \ldots, s_n) = 0 \implies f \text{ is the zero polynomial},$$

[1] This is based on [5], Definition 1.17, but is adapted for our purposes.
[2] [5], Proposition 5.10.

is satisfied. In other words, S is algebraically independent over R_0 if the kernel of the evaluation $\Psi : R_0[X_1, \ldots, X_n] \to R$ at $s_1, \ldots, s_n$ is the zero ideal.

It is worth noting that indeterminates in a ring necessarily form an algebraically independent set.

Lemma 2.12. *Let $S = \{s_1, \ldots, s_n\}$ be algebraically independent over a field k. Then $S' = \{s_1, \ldots, s_{n-1}\}$ is algebraically independent over k.*

Proof. Suppose S' is not algebraically independent over k. Then there exists $f \in k[X_1, \ldots, X_{n-1}]$ such that $f(s_1, \ldots, s_{n-1}) = 0$. But then $g(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-1})$ is such that $g(s_1, \ldots, s_n) = 0$, meaning S is not algebraically independent over k, which is a contradiction. ☐

Lemma 2.13. *Let $S = \{s_1, \ldots, s_n\}$ be algebraically independent over a field k. Then $S' = \{s_1, \ldots, s_{n-2}, s_{n-1}s_n\}$ is algebraically independent over k.*

Proof. Suppose S' is not algebraically independent over k. Then there exists some non-zero polynomial $f \in k[X_1, \ldots, X_{n-1}]$, which we can write as

$$f(X_1, \ldots, X_{n-1}) = \sum_{\lambda \in \Lambda} r_\lambda X_1^{\lambda_1} \cdots X_{n-1}^{\lambda_{n-1}},$$

such that $f(s_1, \ldots, s_{n-2}, s_{n-1}s_n) = 0$. Then the polynomial

$$g(X_1, \ldots, X_n) = \sum_{\lambda \in \Lambda} r_\lambda X_1^{\lambda_1} \cdots X_{n-1}^{\lambda_{n-1}} X_n^{\lambda_{n-1}},$$

is such that $g(s_1, \ldots, s_n) = 0$, which is a contradiction as S is algebraically independent over k. ☐

Lemma 2.14. *Let $n, \tilde{n} \in \mathbb{N}$. If we have a family of pairwise disjoint sets $\{H_i\}_{i=1}^{\tilde{n}}$ where $H_i \subseteq \{1, \ldots, n\}$ for all i, then the function*

$$\eta : \{X_1, \ldots, X_{\tilde{n}}\} \to k[Y_1, \ldots, Y_{n+\tilde{n}}],$$
$$\eta(X_i) = \begin{cases} \prod_{j \in H_i} Y_j & \text{if } H_i \text{ is non-empty,} \\ Y_{n+i} & \text{otherwise,} \end{cases}$$

gives rise to a set $\{\eta(X_i)\}_{i=1}^{\tilde{n}}$ which is algebraically independent over k.

Proof. Because the H_i are disjoint, each indeterminate Y_j divides at most one $\eta(X_i)$. Thus the proof follows from the repeated application of Lemma 2.12 and Lemma 2.13. ☐

Theorem 2.15. *Let $F : W \to \tilde{W}$ be an order-preserving function between 1-dimensional posets with least elements. If F maps the least element of W to the least element of $\tilde{W}$ then there exist rings $D, \tilde{D}$ and a homomorphism $\Phi : \tilde{D} \to D$ such that $W \cong \mathsf{Spec}\, D, \tilde{W} \cong \mathsf{Spec}\, \tilde{D}$ and $\mathsf{Spec}\, \Phi$ is isomorphic to to F.*

[3] Likewise this is based on [5], Definition 1.14.

Proof. Suppose W has $n+1$ elements and $\tilde{W}$ has $\tilde{n}+1$ elements. By Theorem 2.6, the rings

$$D = k[Y_1, \dots, Y_{n+\tilde{n}}]_{\langle Y_1 \rangle, \dots, \langle Y_n \rangle},$$
$$\tilde{D} = k[X_1, \dots, X_{\tilde{n}}]_{\langle X_1 \rangle, \dots, \langle X_{\tilde{n}} \rangle}$$

are such that $W \cong \mathsf{Spec}\, D$ and $\tilde{W} \cong \mathsf{Spec}\, \tilde{D}$. Let w_0 and $\tilde{w}_0$ denote the minimal elements of W and $\tilde{W}$ respectively and arbitrarily label the remaining elements of each poset $w_1, \dots, w_n$ and $\tilde{w}_1, \dots, \tilde{w}_{\tilde{n}}$. We have order isomorphisms

$$\delta(w_i) = \begin{cases} \langle 0 \rangle & \text{if } i = 0, \\ \langle Y_i \rangle & \text{otherwise,} \end{cases} \qquad \tilde{\delta}(\tilde{w}_i) = \begin{cases} \langle 0 \rangle & \text{if } i = 0, \\ \langle X_i \rangle & \text{otherwise.} \end{cases}$$

We introduce the family of sets $\{H_i\}_{i=1}^{\tilde{n}}$, where

$$H_i = \{ j \in \mathbb{N} : F(w_j) = \tilde{w}_i \}.$$

The H_i are pairwise disjoint, so we can use them to define the function η from Lemma 2.14.

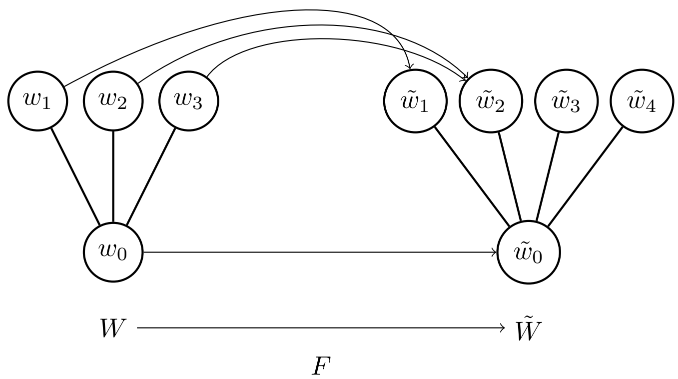
Let $\Psi : k[X_1, \dots, X_{\tilde{n}}] \to k[Y_1, \dots, Y_{n+\tilde{n}}]_{\langle Y_1 \rangle, \dots, \langle Y_n \rangle}$ be the evaluation homomorphism at $\eta(X_1), \dots, \eta(X_{\tilde{n}})$. Let $g \in S = R \setminus \bigcup_{i=1}^{\tilde{n}} \langle X_i \rangle$ and suppose $\Psi(g)$ is a non-unit. Then either $\Psi(g) = 0$ or $\Psi(g) \in \langle Y_i \rangle$ for some $i \in \{1, \dots, n\}$. Since $\eta(Y_1), \dots, \eta(X_{\tilde{n}})$ are algebraically independent over k, $\Psi(g) = 0$ implies g is the zero polynomial and hence a non-unit. If $\Psi(g) \in \langle Y_i \rangle$ then $\Psi(g) = g' Y_i$ for some $g' \in k[Y_1, \dots, Y_{n+\tilde{n}}]$. Hence we must have some X_j such that $Y_i \mid \eta(X_j)$ and $X_j \mid g$. Then $g \in \langle X_j \rangle$ so $g \notin S$. Thus by the universal property of localisation, we have that

$$\Phi : \tilde{D} \to D,$$
$$\Phi\left(\frac{f}{g}\right) = \frac{\Psi(f)}{\Psi(g)},$$

is a well-defined homomorphism.

Finally, we show that Φ is isomorphic to F. Let $\frac{f}{g} \in \Phi^{-1}(\langle 0 \rangle)$. Then $\frac{\Psi(f)}{\Psi(g)} = 0$, and in particular $\Psi(f) = 0$. Since f is the evaluation at an algebraically independent set, this implies f is the zero polynomial, so $\frac{f}{g} = 0$, meaning $\Phi^{-1}(\langle 0 \rangle) = \langle 0 \rangle$. Now consider $\Phi^{-1}(\langle Y_i \rangle)$. Let $\tilde{w}_j = F(w_i)$. Then $i \in H_j$, so $Y_i \mid \eta(X_j)$. Thus $\Phi(X_j) = \eta(X_j) \in \langle Y_i \rangle$, so $X_j \in \Phi^{-1}(\langle Y_i \rangle)$. The only prime ideal of $\tilde{D}$ that contains X_j is $\langle X_j \rangle$, so $F(w_i) = \tilde{w}_j$ implies $\Phi^{-1}(\langle Y_i \rangle) = \langle X_j \rangle$. Now suppose $F(w_i) = \tilde{w}_0$. Suppose $\langle X_j \rangle = \Phi^{-1}(\langle Y_i \rangle)$. Then $\Phi(X_j) = \eta(X_j) \in \langle Y_i \rangle$, but $Y_i \nmid \eta(X_j)$ as $i \notin H_j$ for all j. Therefore $\Phi^{-1}(\langle Y_i \rangle) = \langle 0 \rangle$. $\square$

Example 2.16. Let $F : W \to \tilde{W}$ be the order-preserving function pictured below.



As we can see, W has 4 elements and $\tilde{W}$ has 5, so

$$W \cong \mathsf{Spec}\, k[Y_1, \ldots, Y_7]_{\langle Y_1 \rangle, \ldots, \langle Y_3 \rangle}, \qquad \tilde{W} \cong k[X_1, \ldots, X_4]_{\langle X_1 \rangle, \ldots \langle X_4 \rangle}.$$

We have the sets

$$H_1 = \{1\}, \quad H_2 = \{2, 3\}, \quad H_3 = H_4 = \emptyset,$$

which induce the function

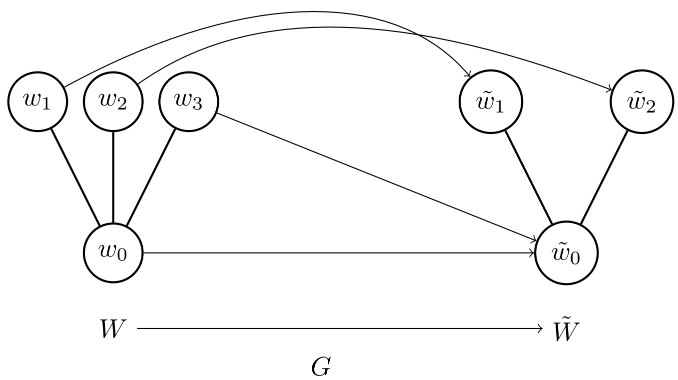
$$\eta : \{X_1, \ldots, X_4\} \to k[Y_1, \ldots, Y_7],$$
$$\eta(X_i) = \begin{cases} Y_1 & \text{if } i = 1, \\ Y_2 Y_3 & \text{if } i = 2, \\ Y_6 & \text{if } i = 3, \\ Y_7 & \text{if } i = 4. \end{cases}$$

Then the homomorphism

$$\Phi : k[X_1, \ldots, X_4]_{\langle X_1 \rangle, \ldots, \langle X_4 \rangle} \to k[Y_1, \ldots, Y_7]_{\langle Y_1 \rangle, \ldots, \langle Y_3 \rangle},$$
$$\Phi\left(\frac{f(X_1, X_2, X_3, X_4)}{g(X_1, X_2, X_3, X_4)}\right) = \frac{f(Y_1, Y_2 Y_3, Y_6, Y_7)}{g(Y_1, Y_2 Y_3, Y_6, Y_7)},$$

induces the order-preserving function $\mathsf{Spec}\, \Phi$ which is isomorphic to F.

Example 2.17. We will repeat the process for the order-preserving function $G : W \to \tilde{W}$ pictured below.



We have

$$W \cong \mathsf{Spec}\, k[Y_1, \dots, Y_5]_{\langle Y_1 \rangle, \dots, \langle Y_3 \rangle}, \qquad \tilde{W} \cong \mathsf{Spec}\, k[X_1, X_2]_{\langle X_1 \rangle, \langle X_2 \rangle}.$$

Then we have the sets

$$H_1 = \{1\}\,, \quad H_2 = \{2\}\,.$$

These sets induce the function

$$\eta : \{X_1, X_2\} \to k[Y_1, \dots, Y_5],$$
$$\eta(X_i) = \begin{cases} Y_1 & \text{if } i = 1, \\ Y_2 & \text{if } i = 2. \end{cases}$$

Then the homomorphism

$$\Phi : k[X_1, X_2]_{\langle X_1 \rangle, \langle X_2 \rangle} \to k[Y_1, \dots, Y_5]_{\langle Y_1 \rangle, \dots, \langle Y_3 \rangle},$$
$$\Phi\left(\frac{f(X_1, X_2)}{g(X_1, X_2)}\right) = \frac{f(Y_1, Y_2)}{g(Y_1, Y_2)},$$

is such that $\mathsf{Spec}\, \Phi$ is isomorphic to G.

Some order-preserving functions do not send the least element of their domain to the least element of their codomain. We give an alternative construction in this case.

Theorem 2.18. *Let $F : W \to \tilde{W}$ be an order-preserving function between 1-dimensional posets with least elements. Then there exist rings $D, \tilde{D}$ and a homomorphism $\Phi : \tilde{D} \to D$ such that $W \cong \operatorname{Spec} D, \tilde{W} \cong \operatorname{Spec} \tilde{D}$ and $\operatorname{Spec} \Phi$ is isomorphic to F.*

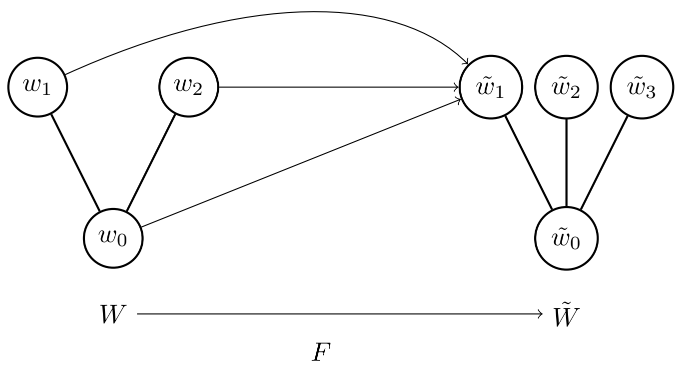
Proof. Let $D, \tilde{D}, \delta, \tilde{\delta}$ be as defined in the previous proof. If F sends the least element of W to the least element of $\tilde{W}$ then the result is proven by Theorem 2.15, so we assume this is not the case. Then for F to be order-preserving, all elements of W must be sent to the same element of $\tilde{W}$. Call this element $\tilde{w}_j$. Let $\Psi : k[X_1, \dots, X_{\tilde{n}}] \to k[Y_1, \dots, Y_{n+\tilde{n}}]_{\langle Y_1 \rangle, \dots, \langle Y_n \rangle}$ be the evaluation at $Y_{n+1}, \dots, Y_{n+j-1}, 0, Y_{n+j+1}, \dots, Y_{n+\tilde{n}}$. Let $g \in S = R \setminus \bigcup_{i=1}^{\tilde{n}}$ and suppose $\Psi(g)$ is a non-unit. Any non-zero polynomial in indeterminates $Y_{n+1}, \dots, Y_{n+j-1}$, $Y_{n+j+1}, \dots, Y_{n+\tilde{n}}$ is a unit of D, so $\Psi(g)$ can only be a non-unit of D if $\Psi(g) = 0$. The kernel of Ψ is $\langle X_j \rangle$, so $\Psi(g) = 0$ implies $g \in \langle X_j \rangle$, in which case $g \notin S$. Therefore by the universal property of localisation, the function

$$\Phi : \tilde{D} \to D,$$
$$\Phi\left(\frac{f}{g}\right) = \frac{\Psi(f)}{\Psi(g)},$$

is a well-defined ring homomorphism.

Finally, we show Φ is isomorphic to F. This is equivalent to showing that $\Phi^{-1}(\mathfrak{p}) = \langle X_j \rangle$ for all $\mathfrak{p} \in \operatorname{Spec} D$. Since $0 \in \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} D$ and $\Phi(X_j) = 0$, we have that $X_j \in \Phi^{-1}(\mathfrak{p})$. The only prime ideal of $\tilde{D}$ that contains X_j is $\langle X_j \rangle$, so $\Phi^{-1}(\mathfrak{p}) = \langle X_j \rangle$. ∎

Example 2.19. Let $F : W \to \tilde{W}$ be the order-preserving function pictured below.



We have $F(w_i) = \tilde{w}_1$ for all i, so the homomorphism

$$\Phi : k[X_1, X_2, X_3]_{\langle X_1 \rangle, \langle X_2 \rangle, \langle X_3 \rangle} \to k[Y_1, \ldots, Y_5]_{\langle Y_1 \rangle, \langle Y_2 \rangle}$$
$$\Phi\left(\frac{f(X_1, X_2, X_3)}{g(X_1, X_2, X_3)}\right) = \frac{f(0, Y_4, Y_5)}{g(0, Y_4, Y_5)}$$

induces the function $\mathsf{Spec}\,\Phi$ which is isomorphic to F.

3 1-Dimensional Posets With Multple Minimal Elements

In this section we split our poset into subposets with least elements, which we can reunite using the 'amalgamated sum'. Using results from the last section, we can find rings isomorphic to these subposets, and connect their spectra using an operation called the 'fibre product' which, by a theorem of Fontana, works analagously to the amalgamated sum, producing a ring with spectrum isomorphic to our original poset.

3.1 Poset Construction

Definition 3.1 (Upper Closure)**.** Let W be a poset and $w \in W$. Then we define the upper closure of w, denoted $w^{\uparrow}$, to be the set of elements greater than or equal to w in the order. That is,

$$w^{\uparrow} = \{w' \in W : w \leq w'\}.$$

Similarly if $X \subseteq W$ then the upper closure of X, denoted $X^{\uparrow}$, is defined as

$$X^{\uparrow} = \{w \in W : x \leq w \text{ for some } x \in X\}.$$

Let $X \subseteq W$. We say X is up-closed if $X = X^{\uparrow}$. Recall that we call a function $g : Z \to Y$ an order-isomorphism if it is bijective and $w_1 \leq w_2$ if and only if $g(w_1) \sqsubseteq g(w_2)$. However, if g has the order-preserving and reflecting properties of an order-isomorphism but is only injective rather than bijective, then we call g an order-embedding. Furthermore, we call g a closed embedding if $g(Z)$ is an up-closed subset of Y.

Definition 3.2 (Amalgamated Sum)**.** Let $(X, \leq_X), (Y, \leq_Y)$ and $Z \subseteq Y$ be posets and $f : Z \to X, g : Z \to Y$ order-preserving functions, where g is a closed embedding. Then the amalgamated sum of X and Y over f and g, denoted $X \sqcup_Z Y$, is the poset $X \sqcup (Y \setminus Z)$ with the order relation

$$\begin{aligned} w_1 \leq w_2 \iff & w_1, w_2 \in X \text{ and } w_1 \leq_X w_2, \text{ or} \\ & w_1, w_2 \in Y \setminus Z \text{ and } w_1 \leq_Y w_2, \text{ or} \\ & w_1 \in Y \setminus Z, w_2 \in X \text{ and } \exists\ z \in Z \text{ s.t. } w_1 \leq_Y g(z) \text{ and } f(z) \leq_X w_2. \end{aligned}$$

For the rest of this section, let W be a 1-dimensional poset with l minimal elements. Our aim is to use properties of W to construct posets X, Y and Z, and then connect these via the amalgamated sum to form a poset isomorphic to W.

We let $w_1, \ldots, w_l$ denote the minimal elements of W, and apply some arbitrary labelling $w_{l+1}, \ldots, w_{|W|}$ to the remaining elements of W. We define the poset

$$V_i = \left\{ v_{i,0}, \ldots, v_{i,|w_i^{\uparrow}|-1} \right\},$$

with the order relation

$$v_{i,j} \leq v_{i,k} \iff j = 0 \text{ or } j = k.$$

Then V_i is a 1-dimensional poset with least element v_0 and $\left|w_i^{\uparrow}\right|$ elements. The same is true for $w_i^{\uparrow}$, so $V_i \cong w_i^{\uparrow}$. We let $\mu_i : V_i \to w_i^{\uparrow}$ be an order isomorphism and let

$$Y = V_1 \sqcup \cdots \sqcup V_l.$$

We define the function

$$\mu : Y \to W,$$
$$\mu(v_{i,j}) = \mu_i(v_{i,j}),$$

which is order-preserving and surjective, but not necessarily injective or order-reflecting.
Some elements of W are contained in multiple sets $w_i^{\uparrow}$, and our aim is to join these elements together to re-form W. Hence, we represent the set of 'joined' maximals of W as

$$J(W) = \bigcup_{i \neq j} w_i^{\uparrow} \cap w_j^{\uparrow}.$$

Now we let $M(W)$ be a set such that

- $J(W) \subseteq M(W) \subset W$, and
- every element of $M(W)$ is a maximal element of W.

We can restrict the set $M(W)$ to $w_i^{\uparrow}$ as follows:

$$M(w_i^{\uparrow}) = M(W) \cap w_i^{\uparrow}.$$

We let

$$Z = \{v_{i,j} \in Y : \mu_i(v_{i,j}) \in M(W)\}.$$

Lemma 3.3. *The poset W can be partitioned into subsets $M(W)$ and $w_i^{\uparrow} \setminus M(w_i^{\uparrow})$ for $i \in \{1, \ldots, l\}$.*

Proof. We have $M(W) \cap (w_i^{\uparrow} \setminus M(w_i^{\uparrow})) = \emptyset$ by definition and $w \in (w_i^{\uparrow} \setminus M(w_i^{\uparrow})) \cap (w_j^{\uparrow} \setminus M(w_j^{\uparrow}))$ for $i \neq j$ implies $w \in w_i^{\uparrow} \cap w_j^{\uparrow} \subseteq M(W)$, which is a contradiction. □

Corollary 3.4. *The surjective restriction of μ_i to $V_i \setminus Z$ is an order isomorphism from $V_i \setminus Z$ to $w_i^{\uparrow} \setminus M(w_i^{\uparrow})$.*

Corollary 3.5. *The surjective restriction of μ to $Y \setminus Z$ is an order isomorphism from $Y \setminus Z$ to $W \setminus M(W)$.*

Now let $X = \{x_1, \ldots, x_{|M(W)|}\}$ be a 0-dimensional poset (i.e. $x_i \leq x_j \iff i = j$). Since $M(W)$ contains only maximal elements of W, it is also a 0-dimensional poset. Since X and $M(W)$ contain the same number of elements, they are isomorphic, so let $\tau : X \to M(W)$ be an order isomorphism. Now define the function

$$f : Z \to X,$$
$$f(v_{i,j}) = \tau^{-1} \circ \mu(v_{i,j}).$$

This function is well-defined by definition of Z, as $v_{i,j} \in Z$ implies $\mu_i(v_{i,j}) \in M(W)$. Let $g : Z \to Y$ be the inclusion function, which is an embedding by definition, and is a closed embedding as all elements of Z are maximal elements of Y. Now we are able to state the following theorem.

Theorem 3.6. *$W \cong X \sqcup_Z Y$.*

Proof. We claim that the function

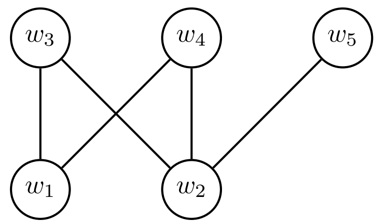
$$\omega : W \to X \sqcup_Z Y,$$
$$\omega(w_j) = \begin{cases} \tau^{-1}(w_j) & \text{if } w_j \in M(W), \\ \mu^{-1}(w_j) & \text{if } w_j \in W \setminus M(W), \end{cases}$$

is an order isomorphism. By Lemma 3.3, W can be partitioned into these different subsets, on which the functions τ^{-1} and μ^{-1} are bijective. Hence ω is itself bijective.

Suppose $w_k \leq w_l$. If $w_k, w_l \in M(W)$ or $w_k, w_l \in W \setminus M(W)$ then $\omega(w_k) \leq \omega(w_l)$. Suppose $w_k \in M(W)$ and $w_l \in W \setminus M(W)$. Then w_k is maximal, so $w_k = w_l$, which is a contradiction. Suppose $w_k \in W \setminus M(W), w_l \in M(W)$. We have $w_k \in w_i^{\uparrow} \setminus M(w_i^{\uparrow})$ for some $i \in \{1, \ldots, l\}$, and also $w_i \leq w_k \leq w_l$, so $w_l \in M(w_i^{\uparrow})$. Let $z = \mu_i^{-1}(w_l)$. Then $f(z) = \tau^{-1}(w_l) = \omega(w_l)$ and $g(z) = \mu_i^{-1}(w_l) \geq \mu_i^{-1}(w_k) = \omega(w_k)$, so $\omega(w_k) \leq \omega(w_l)$.

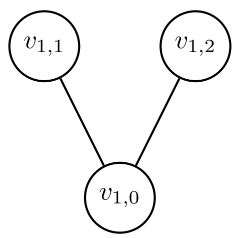
Now suppose $\omega(w_k) \leq \omega(w_l)$. If $\omega(w_k), \omega(w_l) \in X$ or $\omega(w_k), \omega(w_l) \in Y \setminus Z$ then $w_k \leq w_l$, so assume $\omega(w_k) \in Y \setminus Z, \omega(w_l) \in X$ and there exists $z \in Z$ such that $\omega(w_k) \leq g(z)$ and $f(z) \leq \omega(w_l)$. Because g is the inclusion function, we have $\omega(w_k) \leq z$, and since X is 0-dimensional we have $f(z) = \omega(w_l)$. Thus $f(z) = \tau^{-1}(w_l)$, so $\mu(z) = w_l$. We have $\mu^{-1}(w_k) = \omega(w_k) \leq z = \mu^{-1}(w_l)$, and therefore $\mu_i^{-1}(w_k) \leq z\mu_i^{-1}(w_l)$ for some i, so it follows from the order reflecting properties of μ that $w_k \leq w_l$, so ω is an order isomorphism. $\square$

Example 3.7. Let W be the poset below.

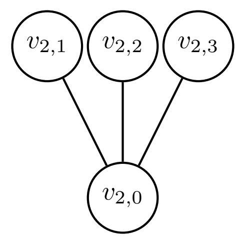


W

Then V_1 and V_2 are as pictured below.



V_1

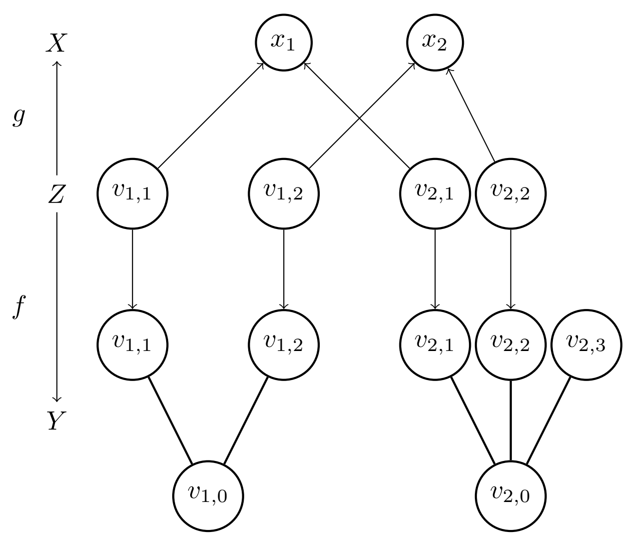


V_2

We have $J(W) = \{w_3, w_4\}$. We define the functions

$$\mu_1(v_{1,i}) = \begin{cases} w_1 & \text{if } i = 0, \\ w_3 & \text{if } i = 1, \\ w_4 & \text{if } i = 2, \end{cases} \qquad \mu_2(v_{1,i}) = \begin{cases} w_2 & \text{if } i = 0, \\ w_3 & \text{if } i = 1, \\ w_4 & \text{if } i = 2, \\ w_5 & \text{if } i = 3, \end{cases}$$

We have $Z = \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}\}$ and $X = \{x_1, x_2\}$ with $\tau(x_1) = w_3, \tau(x_2) = w_4$. Then the functions f and g are as pictured on the next page.


X
g
Z
f
Y
x_1
x_2
$v_{1,1}$
$v_{1,2}$
$v_{2,1}$
$v_{2,2}$
$v_{1,1}$
$v_{1,2}$
$v_{2,1}$
$v_{2,2}$
$v_{2,3}$
$v_{1,0}$
$v_{2,0}$

3.2 Ring Construction

In devising our ring analogue, we use an operation known as the 'fibre product of rings'.

Definition 3.8 (Fibre Product)**.** Let A, B, C be rings and $\phi : A \to C, \psi : B \to C$ be homomorphisms. Then the fibre product of A and B over C, denoted $A \times_C B$, is defined as

$$A \times_C B = \{(a, b) \in A \times B : \phi(a) = \psi(b)\}.$$

A theorem of Fontana found in [4] formalises the way in which the fibre product works analagously to the amalgamated sum.

Theorem 3.9 (Fontana's Theorem)**.** *Let X, Y, Z be posets and $f : Z \to X, g : Z \to Y$ be functions where g is a closed embedding. Let A, B, C be rings and $\phi : A \to C, \psi : B \to C$ be homomorphisms where ψ is surjective. If $\alpha : X \to \mathsf{Spec}\, A$ and $\beta : Y \to \mathsf{Spec}\, B$ are order isomorphisms and $Z \cong \mathsf{Spec}\, C$, and $\mathsf{Spec}\, \phi, \mathsf{Spec}\, \psi$ are isomorphic to f, g respectively, then the function*

$$\chi : X \sqcup_Z Y \to \mathsf{Spec}\, A \times_C B,$$
$$\chi(w) = \begin{cases} p_A^{-1} \circ \alpha(w) & \text{if } w \in X, \\ p_B^{-1} \circ \beta(w) & \text{if } w \in Y \setminus Z, \end{cases}$$

is an order isomorphism, where $p_A : A \times_C B \to A, p_B : A \times_C B \to B$ are projection maps. Furthermore, if $w \in Z$, then

$$p_A^{-1} \circ \alpha \circ f(z) = p_B^{-1} \circ \beta \circ g(z).$$

For $i = 1, \dots, l$, let B_i be a ring such that $\mathsf{Spec}\, B_i \cong V_i$. Such a ring exists by Theorem 2.6, and in fact we have an infinite family of rings to choose from. Then define the family of sets

$$H_i = \{j \in \mathbb{N} : v_{i,j} \in Z\}.$$

These sets codify the way in which we will 'join' the rings. Since each H_i is finite, we order the elements from 1 to $|H_i|$ and use $h_i(j)$ to denote the jth element of H_i. Then we define the ring

$$C_i = B_i / \langle X_{h_i(1)} \rangle \times \cdots \times B_i / \langle X_{h_i(|H_i|)} \rangle.$$

Then the required rings are

$$A = \overbrace{k \times \cdots \times k}^{|X| \text{ times}},$$
$$B = B_1 \times \cdots \times B_l,$$
$$C = C_1 \times \cdots \times C_l.$$

Lemma 3.10. *Let X, Y, Z be as defined in this section's poset construction. Then $X \cong \mathsf{Spec}\, A, Y \cong \mathsf{Spec}\, B$ and $Z \cong \mathsf{Spec}\, C$.*

Proof. The function

$$\alpha : X \to \mathsf{Spec}\, A,$$
$$\alpha(x_i) = R_1 \times \cdots \times \overbrace{\langle 0 \rangle}^{i\text{th place}} \times \cdots \times R_{|W|},$$

is an order isomorphism, as it is a bijection between two 0-dimensional posets. We have $V_i \cong \mathsf{Spec}\, B_i$ for all i, so

$$Y = V_1 \sqcup \cdots \sqcup V_l \cong \mathsf{Spec}\, B_1 \times \cdots \times B_l$$

by Theorem 1.19. In particular, the function

$$\beta : Y \to \mathsf{Spec}\, B,$$
$$\beta(v_{i,j}) = \begin{cases} B_1 \times \cdots \times \overbrace{\langle 0 \rangle}^{i\text{th place}} \times \cdots \times B_l & \text{if } j = 0, \\ B_1 \times \cdots \times \overbrace{\langle X_j \rangle}^{i\text{th place}} \times \cdots \times B_l & \text{if } j \neq 0, \end{cases},$$

is an order isomorphism. By definition of H_i, we have $|Z| = \sum_{i=1}^{l}$, so $Z \cong \mathsf{Spec}\, C$ as C is the product of $|Z|$ fields. The function

$$\gamma : Z \to \mathsf{Spec}\, C,$$
$$\gamma(v_{i,j}) = C_1 \times \cdots \times \overbrace{\langle 1 \rangle \times \cdots \times \underbrace{\langle 0 \rangle}_{k\text{th place}} \times \cdots \times \langle 1 \rangle}^{i\text{th place}} \times \cdots \times C_l, \text{where } h_i(k) = j$$

is bijective, and is therefore an order isomorphism. ☐

The homomorphisms ϕ and ψ that we will use for the fibre product require some results relating to homomorphisms of product rings. The proof that the following lemmas define valid homomorphisms only requires checking that the definition of a homomorphism holds (which can be done in the standard way).

Lemma 3.11. *Let $\phi_i : R \to S_i$ be a homomorphism for $i \in \{1, \ldots, n\}$. Then the function*

$$\phi : R \to S_1 \times \cdots \times S_n,$$
$$\phi(r) = (\phi_1(r), \ldots, \phi_n(r)),$$

is a homomorphism.

Lemma 3.12. *Let $\sigma : \{1, \dots, n\} \to \{1, \dots, m\}$ and let $\phi_i : R_{\sigma(i)} \to S_i$ be a homomorphism for $i \in \{1, \dots, n\}$. Then the function*

$$\phi : R_1 \times \cdots \times R_m \to S_1 \times \cdots \times S_n,$$
$$\phi(r_1, \dots, r_m) = (\phi_1(r_{\sigma(1)}), \dots, \phi_n(r_{\sigma(n)})),$$

is a homomorphism.

Lemma 3.13. *Let X, Y, Z, f, g be as defined in this section's poset construction. Then there exist homomorphisms $\phi : A \to C, \psi : B \to C$ such that $\mathsf{Spec}\,\phi, \mathsf{Spec}\,\psi$ are isomorphic to f and g respectively.*

Proof. Define the function

$$\rho(i, j) = k \iff f(v_{i,j}) = x_k.$$

Now define the homomorphisms

$$\phi_i : A \to C_i,$$
$$\phi_i(a_1, \dots, a_{|X|}) = \left(a_{\rho(i,h_i(1))} + \left\langle X_{h_i(1)} \right\rangle, \dots, a_{\rho(i,h_i(|H_i|))} + \left\langle X_{h_i(|H_i|)} \right\rangle\right),$$
$$\phi : A \to C_i,$$
$$\phi(a_1, \dots, a_{|X|}) = (\phi_1(a_1, \dots, a_{|X|}), \cdots, \phi_l(a_1, \dots, a_{|X|})).$$

The function ϕ is a homomorphism by Lemma 3.11, and we claim that $\mathsf{Spec}\,\phi$ is isomorphic to f. We must show that $\alpha \circ f(v_{i,j}) = \phi^{-1} \circ \gamma(v_{i,j})$. We have $f(v_{i,j}) = x_{\rho(i,j)}$, so we must show that $\phi^{-1} \circ \gamma(v_{i,j}) = \alpha(x_{\rho(i,j)})$. We have

$$\gamma(v_{i,j}) = C_1 \times \cdots \times \overbrace{\langle 1 \rangle \times \cdots \times \underbrace{\langle 0 \rangle}_{k\text{th place}} \times \cdots \times \langle 1 \rangle}^{i\text{th place}} \times \cdots \times C_l, \text{where } h_i(k) = j.$$

Suppose $(a_1, \dots, a_{|X|}) \in \phi^{-1} \circ \gamma(v_{i,j})$. Then

$$\begin{aligned}
&\phi_i(a_1, \dots, a_{|X|}) \\
&\quad = (a_{\rho(i,h_i(1))} + \left\langle X_{h_i(1)} \right\rangle, \dots, a_{\rho(i,h_i(k))} + \left\langle X_{h_i(k)} \right\rangle, \dots, a_{\rho(i,h_i(|H_i|))} + \left\langle X_{h_i(|H_i|)} \right\rangle) \\
&\quad \in \langle 1 \rangle \times \cdots \underbrace{\langle 0 \rangle}_{k\text{th place}} \times \cdots \times \langle 1 \rangle,
\end{aligned}$$

so $a_{\rho(i,h_i(k))} \in \left\langle X_{h_i(k)} \right\rangle$, meaning $a_{\rho(i,h_i(k))} = 0$. Note that

$$r = (1, \dots, \overbrace{0}^{l\text{th place}}, \dots, 1) \in \alpha(x_l)$$

for all l, but $r \notin \phi^{-1} \circ \gamma(v_{i,j})$ if $l \neq \rho(i, h_i(k)) = \rho(i, j)$. Hence $\phi^{-1} \circ \gamma(v_{i,j}) = \alpha(x_{\rho(i,j)})$ as required.

Define the homomorphisms

$$\psi_i : B_i \to C_i,$$
$$\psi_i\left(\frac{f}{g}\right) = \left(\frac{f}{g} + \left\langle X_{h_i(1)}\right\rangle, \dots, \frac{f}{g} + \left\langle X_{h_i(|H_i|)}\right\rangle\right),$$
$$\psi\left(\frac{f_1}{g_1}, \dots, \frac{f_l}{g_l}\right) = \left(\phi_1\left(\frac{f_1}{g_1}\right), \dots, \phi_l\left(\frac{f_l}{g_l}\right)\right).$$

The function ψ is a homomorphism by Lemma 3.12. The surjectivity of ψ_i follows from the Chinese Remainder Theorem for rings and maximality of each ideal $\langle X_j\rangle$, and surjectivity of ψ follows directly from surjectivity of each ψ_i. We claim that $\mathsf{Spec}\,\psi$ is isomorphic to g. We have $g(v_{i,j}) = v_{i,j}$ as g is the inclusion function. Moreover $v_{i,0} \notin Z$ for all i, so our aim is to show that

$$\psi^{-1}(\gamma(v_{i,j})) = B_1 \times \cdots \times \overbrace{\langle X_j\rangle}^{i\text{th place}} \times \cdots \times B_l.$$

Suppose $\left(\frac{f_1}{g_1}, \dots, \frac{f_l}{g_l}\right) \in \psi^{-1}(\gamma(v_{i,j}))$. Then

$$\begin{aligned}&\psi_i\left(\frac{f_i}{g_i}\right)\\ &= \left(\frac{f_i}{g_i} + \left\langle X_{h_i(1)}\right\rangle, \dots, \frac{f_i}{g_i} + \left\langle X_{h_i(k)}\right\rangle, \dots, \frac{f_i}{g_i} + \left\langle X_{h_i(|H_i|)}\right\rangle\right)\\ &\in \overbrace{\langle 1\rangle \times \cdots \times \underbrace{\langle 0\rangle}_{k\text{th place}} \times \cdots \times \langle 1\rangle}^{i\text{th place}}, \text{where } h_i(k) = j,\end{aligned}$$

so $\frac{f_i}{g_i} + \left\langle X_{h_i(k)}\right\rangle = \frac{f_i}{g_i} + \langle X_j\rangle = 0 + \langle X_j\rangle$. Hence $\frac{f_i}{g_i} \in \langle X_j\rangle$. We have

$$r = (1, \dots, \overbrace{0}^{l\text{th place}}, \dots, 1) \in \beta(v_{l,m})$$

for $l \neq i$, but $r \notin \psi^{-1}(\gamma(v_{i,j}))$. Hence $\psi^{-1}(\gamma(v_{i,j})) = \beta(v_{i,m})$ for some m. But note that

$$r = (1, \dots, \overbrace{X_j}^{i\text{th place}}, \dots, 1) \notin \beta(v_{i,m})$$

for $m \neq j$, but $r \in \psi^{-1}(\gamma(v_{i,j}))$. Hence $\psi^{-1}(\gamma(v_{i,j})) = \beta(v_{i,j})$ as required. □

The last two results combined with Fontana's theorem tell us that $\mathsf{Spec}\,A \times_C B \cong X \sqcup_Z Y \cong W$. Let $\delta : W \to \mathsf{Spec}\,A \times_C B, \delta(w) = \chi \circ \omega(w)$, or explicitly,

$$\delta(w) = \begin{cases} p_A^{-1} \circ \alpha \circ \tau^{-1}(w) & \text{if } w \in M(W), \\ p_B^{-1} \circ \beta \circ \mu^{-1}(w) & \text{if } w \in W \setminus M(W). \end{cases}$$

Example 3.14. Let W be the poset from Example 3.7. We have $H_1 = H_2 = \{1, 2\}$ with $h_1(1) = h_2(1) = 1$ and $h_1(2) = h_2(2) = 2$. Then the required rings are

$$\begin{aligned}
A &= k \times k, \\
B_1 &= k[X_1, X_2]_{\langle X_1 \rangle, \langle X_2 \rangle}, \\
B_2 &= k[X_1, X_2, X_3]_{\langle X_1 \rangle, \langle X_2 \rangle, \langle X_3 \rangle}, \\
B &= B_1 \times B_2 \\
C &= B_1 / \langle X_1 \rangle \times B_1 / \langle X_2 \rangle \times B_2 / \langle X_1 \rangle \times B_2 / \langle X_2 \rangle .
\end{aligned}$$

We have $\rho(1, 1) = \rho(2, 1) = 1$ and $\rho(1, 2) = \rho(2, 2) = 2$. The required homomorphisms are

$$\begin{gathered}
\phi : A \to C, \\
\phi(a_1, a_2) = (a_1 + \langle X_1 \rangle, a_2 + \langle X_2 \rangle, a_1 + \langle X_1 \rangle, a_2 + \langle X_2 \rangle), \\
\psi : B \to C, \\
\psi\left(\frac{f_1}{g_1}, \frac{f_2}{g_2}\right) = \left(\frac{f_1}{g_1} + \langle X_1 \rangle, \frac{f_1}{g_1} + \langle X_2 \rangle, \frac{f_2}{g_2} + \langle X_1 \rangle, \frac{f_2}{g_2} + \langle X_2 \rangle\right).
\end{gathered}$$

Then $W \cong \mathsf{Spec}\, A \times_C B$.

3.3 Homomorphism Construction

Let $W, \tilde{W}$ be 1-dimensional posets. We refer to objects used in the poset and ring constructions of $\tilde{W}$ using tildes. In this section we give a construction for a homomorphism corresponding to an order-preserving function $F : W \to \tilde{W}$, provided we enforce an extra limitation. The limitation in question requires that we can find a set $M(\tilde{W})$ as described in the poset construction procedure such that $F(J(W)) \subseteq M(\tilde{W})$.

We assume that such a set $M(\tilde{W})$ exists, and use it in the poset and ring construction procedures to produce a ring $\tilde{D}$ with spectrum isomorphic to $\tilde{W}$. We choose rings $\tilde{B}_i$ with no throw-away indeterminates. We take $M(W) = J(W)$ in the poset and ring construction processes to produce D, and in choosing rings B_i we add throw-away indeterminates. The number of throw-away indeterminates we add will be determined later in the process.

Since $F(M(W)) \subseteq M(\tilde{W})$, the function

$$F_A : X \to \tilde{X},$$
$$F_A(x_i) = \tilde{\tau}^{-1} \circ F \circ \tau(x_i),$$

is well-defined, and by definition is isomorphic to the restriction of the domain of F to $M(W)$. Now define the function

$$\theta : \{1, \dots, |X|\} \to \left\{1, \dots, \left|\tilde{X}\right|\right\},$$
$$\theta(i) = j \iff F_A(x_i) = \tilde{x}_j.$$

Then define the homomorphism

$$\Phi_A : \tilde{A} \to A,$$
$$\Phi_A(a_1, \dots, a_{|\tilde{X}|}) = (a_{\theta(1)}, \dots, a_{\theta(|X|)}).$$

This is well-defined by applying Lemma 3.11 to the projections of $\tilde{A}$ to its subfields k.

Proposition 3.15. *F_A is isomorphic to* $\mathsf{Spec}\,\Phi_A$*.*

Proof. Suppose $F_A(x_i) = \tilde{x}_j$. Note that $\theta(i) = j$. We want to show that $\Phi_A^{-1} \circ \alpha(x_i) = \tilde{\alpha}(\tilde{x}_j)$. Suppose $(a_1, \dots, a_{|\tilde{X}|}) \in \Phi_A^{-1} \circ \alpha(x_i)$. Then

$$\begin{aligned} &\Phi_A(a_1, \dots, a_{|\tilde{X}|}) \\ &\quad = (a_{\theta(1)}, \dots, a_{\theta(i)}, \dots, a_{\theta(|X|)}) \\ &\quad \in k \times \cdots \times \overbrace{\langle 0 \rangle}^{i\text{th place}} \times \cdots \times k, \end{aligned}$$

so $a_{\theta(i)} = a_j = 0$. Note that

$$r = (1, \dots, \overbrace{0}^{k\text{th place}}, \dots, 1) \in \tilde{\alpha}(\tilde{x}_k),$$

but $r \notin \Phi_A^{-1} \circ \alpha(x_i)$ for $k \neq j$. The only remaining possibility is $\Phi_A^{-1} \circ \alpha(x_i) = \tilde{\alpha}(\tilde{x}_j)$. □

Suppose $F(w_i) = \tilde{w}_k$ for $i \in \{1, \dots, l\}$. There exists some $j \in \left\{1, \dots, \tilde{l}\right\}$ such that $\tilde{w}_k \in \tilde{w}_j^{\uparrow}$. Then since F is order presering, for all $w_l \in w_i^{\uparrow}$ we have $F(w_l) \geq F(w_i) = \tilde{w}_k \geq \tilde{w}_j$, so $F(w_i^{\uparrow}) \subseteq \tilde{w}_j^{\uparrow}$ for some $j \in \left\{1, \dots, \tilde{l}\right\}$. Hence the function

$$\sigma : \{1, \dots, l\} \to \left\{1, \dots, \tilde{l}\right\},$$
$$\sigma(i) = j \implies F(w_i^{\uparrow}) \subseteq \tilde{w}_j^{\uparrow},$$

is well-defined (note that the function is not necessarily unique). Then we have the induced functions

$$F_i : V_i \to \tilde{V}_{\sigma(i)},$$
$$F_i(v_{i,j}) = \tilde{\mu}_{\sigma(i)}^{-1} \circ F \circ \mu_i(v_{i,j}),$$

which by definition are isomorphic to the restrictions of the domain of F to $w_i^{\uparrow}$. Each of these is an order-preserving function between 1-d posets with least elements, so by Theorem 2.18 there exist homomorphisms $\Phi_i : \tilde{B}_{\sigma(i)} \to B_i$ such that $\mathsf{Spec}\,\Phi_i$ is isomorphic to F_i. This means that, if $\mathfrak{p} \in \mathsf{Spec}\,B$ and $\mathfrak{q} \in \mathsf{Spec}\,\tilde{B}$ and

$$\beta(v_{i,j}) = B_1 \times \cdots \times \mathfrak{p} \times \cdots \times B_l,$$
$$\tilde{\beta}(\tilde{v}_{\sigma(i),k}) = \tilde{B}_1 \times \cdots \times \mathfrak{q} \times \cdots \times \tilde{B}_{\tilde{l}}$$

then $F_i(v_{i,j}) = \tilde{v}_{\sigma(i),k}$ if and only if $\Phi_i^{-1}(\mathfrak{p}) = \mathfrak{q}$. To allow for these homomorphisms to be constructed, we choose the number of throw-away indeterminates in B_i to be the number of indeterminates in $\tilde{B}_{\sigma(i)}$. Define the function

$$F_B : Y \to \tilde{Y},$$
$$F_B(v_{i,j}) = F_i(v_{i,j}).$$

Now define the function

$$\Phi_B : \tilde{B} \to B,$$
$$\Phi_B\left(\frac{f_1}{g_1}, \dots, \frac{f_{\tilde{l}}}{g_{\tilde{l}}}\right) = \left(\Phi_1\left(\frac{f_{\sigma(1)}}{g_{\sigma(1)}}\right), \dots, \Phi_l\left(\frac{f_{\sigma(l)}}{g_{\sigma(l)}}\right)\right).$$

Our aim is to show that $\mathsf{Spec}\,\Phi_B$ is isomorphic to F_B, and we require the following lemma, which uses the projection homomorphisms $p_i : B \to B_i$.

Lemma 3.16. [4] *Let* $\mathfrak{p} \in \operatorname{Spec} B_i$. *Then* $\tilde{p}_{\sigma(i)}^{-1} \circ \Phi_i^{-1}(\mathfrak{p}) = \Phi_B^{-1} \circ p_i^{-1}(\mathfrak{p})$.

Proof. Let $(r_1, \dots, r_{\tilde{l}}) \in \tilde{p}_{\sigma(i)}^{-1} \circ \Phi_i^{-1}(\mathfrak{p})$. Then $r_{\sigma(i)} \in \Phi_i^{-1}(\mathfrak{p})$, so there exists $s_i \in \mathfrak{p}$ such that $\Phi_i(r_{\sigma(i)})) = s_i$. Since $s_j = \Phi_j(r_{\sigma(j)}) \in B_j$ for all j, we have $(s_1, \dots, s_i, \dots, s_l) \in p_i^{-1}(\mathfrak{p})$. Note that $\Phi_B(r_1, \dots, r_{\tilde{l}}) = (s_1, \dots, s_i, \dots, s_l)$, so $(r_1, \dots, r_{\tilde{l}}) \in \Phi_B^{-1} \circ p_i^{-1}(\mathfrak{p})$.
Now let $(r_1, \dots, r_{\tilde{l}}) \in \Phi_B^{-1} \circ p_i^{-1}(\mathfrak{p})$. Then there exists $(s_1, \dots, s_l) \in p_i^{-1}(\mathfrak{p})$ such that $\Phi_B(r_1, \dots, r_{\tilde{l}}) = (s_1, \dots, s_l)$. Then $s_i \in \mathfrak{p}$, and since $\Phi_i(r_{\sigma(i)}) = s_i$, we have $r_{\sigma(i)} \in \Phi_i^{-1}(\mathfrak{p})$, so $(r_1, \dots, r_{\tilde{l}}) \in \tilde{p}_{\sigma(i)}^{-1} \circ \Phi_i^{-1}(\mathfrak{p})$. □

Proposition 3.17. F_B *is isomorphic to* $\operatorname{Spec} \Phi_B$.

Proof. Suppose $F(v_{i,j}) = \tilde{v}_{\sigma(i),k}$ and

$$\begin{aligned} \beta(v_{i,j}) &= B_1 \times \cdots \times \mathfrak{p} \times \cdots \times B_l, \\ \tilde{\beta}(\tilde{v}_{\sigma(i),k}) &= \tilde{B}_1 \times \cdots \times \mathfrak{q} \times \cdots \times \tilde{B}_{\tilde{l}}. \end{aligned}$$

Note that $\beta(v_{i,j}) = p_i^{-1}(\mathfrak{p})$ and $\tilde{\beta}(v_{\sigma(i),k}) = \tilde{p}_{\sigma(i)}^{-1}(\mathfrak{q})$. Then we have

$$\begin{aligned} \tilde{\beta}^{-1} \circ \Phi_B^{-1} \circ \beta(v_{i,j}) &= \tilde{\beta}^{-1} \circ \Phi_B^{-1} \circ p_i^{-1}(\mathfrak{p}) \\ &= \tilde{\beta}^{-1} \circ \tilde{p}_{\sigma(i)}^{-1} \circ \Phi_i^{-1}(\mathfrak{p}) \\ &= \tilde{\beta}^{-1} \circ \tilde{p}_{\sigma(i)}^{-1}(\mathfrak{q}) \\ &= \tilde{v}_{\sigma(i),k}, \end{aligned}$$

as required. □

We shall combine Φ_A and Φ_B in a similar fashion to Lemma 3.12 to form a function Φ, but since we are not dealing with regular product rings, we must first check that the 'fibre product constraint' is always satisfied by elements in the image of this function.

Proposition 3.18. *Let* $(a, b) \in \tilde{D}$. *Then* $(\Phi_A(a), \Phi_B(b)) \in D$.

Proof. If $(\Phi_A(a), \Phi_B(b))$ then $\phi(\Phi_A(a)) = \psi(\Phi_B(b))$, which means $\phi_i \circ \Phi_A(a) = \psi_i \circ \Phi_i \left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right)$, which can be equivalently stated as

$$\Phi_i \left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right) + \left\langle Y_{h_i(j)} \right\rangle = a_{\theta \circ \rho(i, h_i(j))} + \left\langle Y_{h_i(j)} \right\rangle$$

for $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, |H_i|\}$.

First, we aim to show that if $F_B(v_{i,h_i(j)}) = \tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}$ then $\theta \circ \rho(i, h_i(j)) =$

[4] We use several results of this type in this section, and use a result similar to it in the next. The results come from the fact that we can form a 'commutative square' with the rings and homomorphisms involved.

$\tilde{\rho}(\sigma(i), \tilde{h}_{\sigma(i)}(k))$. Suppose $\rho(i, h_i(j)) = r$ and $\tilde{\rho}(\sigma(i), \tilde{h}_{\sigma(i)}(k)) = s$. Then $f(v_{i,h_i(j)}) = \tau^{-1} \circ \mu(v_{i,h_i(j)}) = x_r$ and $\tilde{f}(\tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}) = \tilde{\tau}^{-1} \circ \tilde{\mu}(\tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}) = \tilde{x}_s$. Then we have

$$\begin{aligned}
F_A(x_r) &= \tilde{\tau}^{-1} \circ F \circ \tau(x_r) \\
&= \tilde{\tau}^{-1} \circ F \circ \tau \circ \tau^{-1} \circ \mu(v_{i,h_i(j)}) \\
&= \tilde{\tau}^{-1} \circ F \circ \mu(v_{i,h_i(j)}) \\
&= \tilde{\tau}^{-1} \circ \tilde{\mu} \circ \tilde{\mu}^{-1} F \circ \mu(v_{i,h_i(j)}) \\
&= \tilde{\tau}^{-1} \circ \tilde{\mu} \circ F_B(v_{i,h_i(j)}) \\
&= \tilde{\tau}^{-1} \circ \tilde{\mu}(\tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}) \\
&= \tilde{x}_s,
\end{aligned}$$

so $\theta(r) = s$. Hence $\theta \circ \rho(i, h_i(j)) = \tilde{\rho}(\sigma(i), \tilde{h}_{\sigma(i)}(k))$.

Because $(a, b) \in D$, we have $\tilde{\phi}(a) = \tilde{\psi}(b)$, so $\tilde{\psi}_{\sigma(i)}\left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right) = \tilde{\phi}_{\sigma(i)}(a)$. In particular, we have

$$\frac{f_{\sigma(i)}(X_1, \ldots, X_{|\tilde{B}_{\sigma(i)}|-1})}{g_{\sigma(i)}(X_1, \ldots, X_{|\tilde{B}_{\sigma(i)}|-1})} + \left\langle X_{\tilde{h}_{\sigma(i)}(k)} \right\rangle = a_{\tilde{\rho}(\sigma(i),\tilde{h}_{\sigma(i)}(k))} + \left\langle X_{\tilde{h}_{\sigma(i)}(k)} \right\rangle.$$

Equivalently, there exists $\frac{p}{q} \in \tilde{B}_{\sigma(i)}$ such that

$$\frac{f_{\sigma(i)}(X_1, \ldots, X_{|\tilde{B}_{\sigma(i)}|-1})}{g_{\sigma(i)}(X_1, \ldots, X_{|\tilde{B}_{\sigma(i)}|-1})} = \frac{p}{q} X_{\tilde{h}_{\sigma(i)}(k)} + a_{\tilde{\rho}(\sigma(i),\tilde{h}_{\sigma(i)}(k))}.$$

Recall that $F_B(v_{i,h_i(j)}) = \tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}$. If $\mathsf{Spec}\,\Phi_i$ sends the minimal element of $\mathsf{Spec}\,B_i$ to the minimal element of $\mathsf{Spec}\,\tilde{B}_{\sigma(i)}$ then $Y_{h_i(j)} \mid \eta(X_{\tilde{h}_{\sigma(i)}(k)})$, and we have

$$\begin{aligned}
&\Phi_i\left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right) + \langle Y_{h_i(j)} \rangle = \frac{f_{\sigma(i)}(\eta(X_1), \ldots, \eta(X_{|\tilde{B}_{\sigma(i)}|-1}))}{g_{\sigma(i)}(\eta(X_1), \ldots, \eta(X_{|\tilde{B}_{\sigma(i)}|-1}))} + \langle Y_{h_i(j)} \rangle \\
&\quad = \frac{p(\eta(X_1), \ldots, \eta(X_{|\tilde{B}_{\sigma(i)}|-1}))}{q(\eta(X_1), \ldots, \eta(X_{|\tilde{B}_{\sigma(i)}|-1}))} \eta(X_{\tilde{h}_{\sigma(i)}(k)}) + a_{\tilde{\rho}(\sigma(i),\tilde{h}_{\sigma(i)}(k))} + \langle Y_{h_i(j)} \rangle \\
&\quad = a_{\theta \circ \rho(i,h_i(j))} + \langle Y_{h_i(j)} \rangle.
\end{aligned}$$

Now suppose $\mathsf{Spec}\,\Phi_i$ sends the minimal element of $\mathsf{Spec}\,B_i$ to a maximal element of $\mathsf{Spec}\,\tilde{B}_{\sigma(i)}$. To preserve order, all prime ideals must be sent to the same place. We have $F_B(v_{i,h_i(h)}) = \tilde{v}_{\sigma(i),\tilde{h}_{\sigma(i)}(k)}$, so $\Phi^{-1}(\langle Y_{h_i(j)} \rangle) = \left\langle X_{\tilde{h}_{\sigma(i)}(k)} \right\rangle$, hence all prime ideals must be sent to $\left\langle X_{\tilde{h}_{\sigma(i)}(k))} \right\rangle$, so $X_{\tilde{h}_{\sigma(i)}(k)}$ is evaluated at zero in the homomorphism. Therefore

$$
\begin{aligned}
&\Phi_i\left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right) + \left\langle Y_{h_i(j)}\right\rangle \\
&= \frac{f_{\sigma(i)}(\eta(X_1),\ldots,\overbrace{0}^{\tilde{h}_{\sigma(i)}(k)\text{th place}},\ldots,\eta(X_{|\tilde{B}_{\sigma(i)}|-1}))}{g_{\sigma(i)}(\eta(X_1),\ldots,\underbrace{0}_{\tilde{h}_{\sigma(i)}(k)\text{th place}},\ldots,\eta(X_{|\tilde{B}_{\sigma(i)}|-1}))} + \left\langle Y_{h_i(j)}\right\rangle \\
&= \frac{p}{q}\cdot 0 + a_{\tilde{\rho}(\sigma(i),\tilde{h}_{\sigma(i)}(k))} + \left\langle Y_{h_i(j)}\right\rangle \\
&= a_{\theta\circ\rho(i,h_i(j))} + \left\langle Y_{h_i(j)}\right\rangle .
\end{aligned}
$$

□

Finally, we can define the homomorphism

$$
\begin{aligned}
\Phi &: \tilde{D} \to D, \\
\Phi(a,b) &= (\Phi_A(a), \Phi_B(b)).
\end{aligned}
$$

The following pair of lemmas makes it easy to prove that $\mathsf{Spec}\,\Phi$ is isomorphic to F. They involve the use of the projection maps

$$
\begin{aligned}
p_A &: A \times_C B \to A, \\
p_B &: A \times_C B \to B, \\
p_{\tilde{A}} &: \tilde{A} \times_{\tilde{C}} \tilde{B} \to \tilde{A}, \\
p_{\tilde{B}} &: \tilde{A} \times_{\tilde{C}} \tilde{B} \to \tilde{B}.
\end{aligned}
$$

Lemma 3.19. $\Phi^{-1} \circ p_A^{-1} = p_{\tilde{A}}^{-1}\Phi_A^{-1}$.

Proof. Let $\mathfrak{p} \in \mathsf{Spec}\,\tilde{A} \times_{\tilde{C}} \tilde{B}$. Let $(a,b) \in \Phi^{-1} \circ p_A^{-1}(\mathfrak{p})$. Then there exists $(c,d) \in p_A^{-1}(\mathfrak{p})$ such that $\Phi(c,d) = (\Phi_A(c), \Phi_B(d)) = (a,b)$. Then $c \in \mathfrak{p}$, so $a \in \Phi_A^{-1}(\mathfrak{p})$, and $(a,b) \in \tilde{A} \times_{\tilde{C}} \tilde{B}$ so $(a,b) \in p_{\tilde{A}}^{-1}\Phi_A^{-1}$.

Let $(a,b) \in p_{\tilde{A}}^{-1}\Phi_A^{-1}(\mathfrak{p})$. Then $a \in \Phi_A^{-1}(\mathfrak{p})$, so there exists $c \in \mathfrak{p}$ such that $\Phi_A(a) = c$. Recall that if $(a,b) \in \tilde{A} \times_{\tilde{C}} \tilde{B}$ then $(\Phi_A(a), \Phi_B(b)) \in A \times_C B$, so let $d = \Phi_B(b)$. Then $(c,d) \in p_A^{-1}(\mathfrak{p})$, so $(a,b) \in \Phi^{-1} \circ p_A^{-1}(\mathfrak{p})$. □

The proof of the following lemma is near-identical.

Lemma 3.20. $\Phi^{-1} \circ p_B^{-1} = p_{\tilde{B}}^{-1}\Phi_B^{-1}$.

Theorem 3.21. $\mathsf{Spec}\,\Phi$ *is isomorphic to* F.

Proof. We must show that $F(w) = \tilde{\delta} \circ \Phi^{-1} \circ \delta(w)$ for all $w \in W$. If $w \in M(W)$ then let $x_j = \tau^{-1}(w)$ and $\tilde{x}_k = \tilde{\tau}^{-1} \circ F(w)$. Then we have

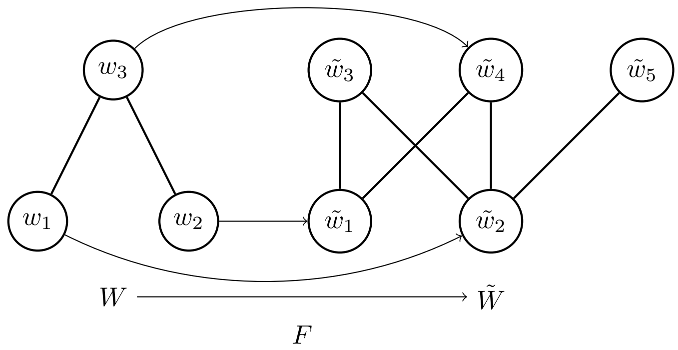
$$\begin{aligned}
\tilde{\delta} \circ \Phi^{-1} \circ \delta(w) &= \tilde{\tau} \circ \tilde{\alpha}^{-1} \circ p_{\tilde{A}} \circ \Phi^{-1} \circ p_A^{-1} \circ \alpha \circ \tau^{-1}(w) \\
&= \tilde{\tau} \circ \tilde{\alpha}^{-1} \circ p_{\tilde{A}} \circ p_{\tilde{A}}^{-1} \circ \Phi_A^{-1} \circ \alpha \circ \tau^{-1}(w) \\
&= \tilde{\tau} \circ \tilde{\alpha}^{-1} \circ \Phi_A^{-1} \circ \alpha \circ \tau^{-1}(w) \\
&= \tilde{\tau} \circ F_A \circ \tau^{-1}(w) \\
&= F(w).
\end{aligned}$$

If $w \in W$ then there exists $v_{i,j} \in Y$ such that $\mu(v_{i,j}) = w$, and $\tilde{v}_{\sigma(i),k}$ such that $\tilde{\mu}(\tilde{v}_{\sigma(i),k}) = F(w)$. Then we have

$$\begin{aligned}
\tilde{\delta} \circ \Phi^{-1} \circ \delta(w) &= \tilde{\tau} \circ \tilde{\beta}^{-1} \circ p_{\tilde{B}} \circ \Phi^{-1} \circ p_B^{-1} \circ \beta \circ \mu^{-1}(w) \\
&= \tilde{\mu} \circ \tilde{\beta}^{-1} \circ p_{\tilde{B}} \circ p_{\tilde{B}}^{-1} \circ \Phi_B^{-1} \circ \beta \circ \mu^{-1}(w) \\
&= \tilde{\mu} \circ \tilde{\beta}^{-1} \circ \Phi_B^{-1} \circ \beta \circ \mu^{-1}(w) \\
&= \tilde{\mu} \circ F_B \circ \mu^{-1}(w) \\
&= F(w).
\end{aligned}$$

□

Example 3.22. Consider the order-preserving function below.



We constructed a ring $\tilde{D}$ with spectrum isomorphic to $\tilde{W}$ in Example 3.14. We have that $W \cong X \sqcup_Z Y$ where $X = \{x_1\}, Y = \{v_{1,0}, v_{1,1}, v_{2,0}, v_{2,1}\}, Z = \{v_{1,1}, v_{2,1}\}$ and $f(v_{1,1}) = f(v_{2,1}) = x_1$. Furthermore if

$$\begin{aligned}
A &= k, \\
B_1 &= k[Y_1, Y_2, Y_3, Y_4]_{\langle Y_1 \rangle}, \\
B_2 &= k[Y_1, Y_2, Y_3]_{\langle Y_1 \rangle}, \\
B &= B_1 \times B_2, \\
C &= B_1 / \langle Y_1 \rangle \times B_2 / \langle Y_1 \rangle
\end{aligned}$$

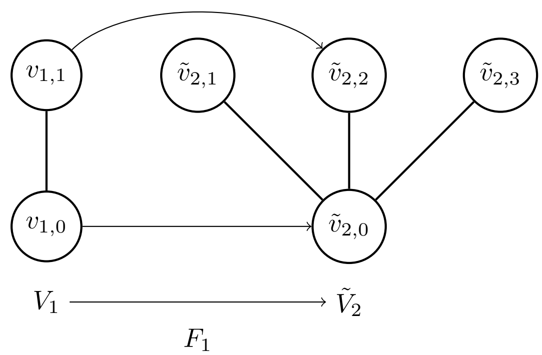
and ϕ, ψ are the homomorphisms obtained from the ring construction and $D = A \times_C B$, then $W \cong \operatorname{Spec} D$. We have $\sigma(1) = 2, \sigma(2) = 1, F_A(x_1) = x_2, \theta(1) = 2$ and the functions F_1 and F_2 are as pictured on the next page. Then we have $\Phi_A(a_1, a_2) = a_2$,

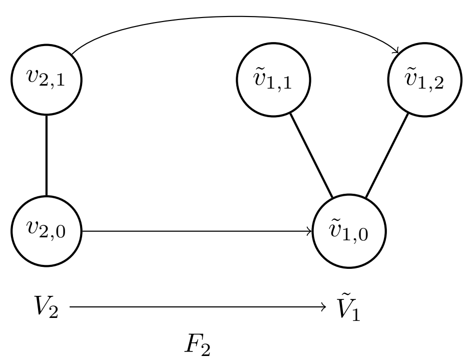
$$\begin{gathered}
\Phi_1 : \tilde{B}_2 \to B_1, \\
\Phi_1 \left(\frac{f(X_1, X_2, X_3)}{g(X_1, X_2, X_3)} \right) = \frac{f(Y_2, Y_1, Y_4)}{g(Y_2, Y_1, Y_4)}, \\
\Phi_2 : \tilde{B}_1 \to B_2, \\
\Phi_2 \left(\frac{f(X_1, X_2)}{g(X_1, X_2)} \right) = \frac{f(Y_2, Y_1)}{g(Y_2, Y_1)},
\end{gathered}$$

so the function

$$\begin{gathered}
\Phi : \tilde{D} \to D, \\
\Phi \left((a_1, a_2), \left(\frac{f_1(X_1, X_2)}{g_1(X_1, X_2)}, \frac{f_2(X_1, X_2, X_3)}{g_2(X_1, X_2, X_3)} \right) \right) = \left(a_2, \left(\frac{f_2(Y_2, Y_1, Y_4)}{g_2(Y_2, Y_1, Y_4)}, \frac{f_1(Y_2, Y_1)}{g_1(Y_2, Y_1)} \right) \right)
\end{gathered}$$

is such that $\operatorname{Spec} \Phi$ is isomorphic to F.


$v_{1,1}$
$\tilde{v}_{2,1}$
$\tilde{v}_{2,2}$
$\tilde{v}_{2,3}$
$v_{1,0}$
$\tilde{v}_{2,0}$
V_1
$\tilde{V}_2$
F_1


$v_{2,1}$
$\tilde{v}_{1,1}$
$\tilde{v}_{1,2}$
$v_{2,0}$
$\tilde{v}_{1,0}$
V_2
$\tilde{V}_1$
F_2

4 n-Dimensional 'Trees'

In this section, we give a final set of poset, ring and homomorphism construction methods. This time, the methods will be used to construct rings isomorphic to posets of higher dimensions. In contrast to the separate gluing and joining procedures seen in other methods, our restriction of this method to a class of posets resembling 'upside-down trees' allows us to both 'glue' and 'join' posets using a single application of the amalgamated sum, and hence a single application of the fibre product.

Definition 4.1 (Tree)**.** Let W be a poset. We say W is a tree if it is connected, has a greatest element, and $w^{\uparrow}$ is totally-ordered for every $w \in W$.

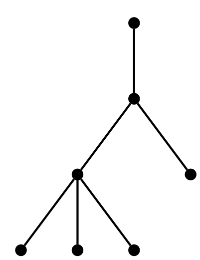


Figure 1: A 3-dimensional tree.

Note that in some texts, a tree is defined to have $w^{\downarrow}$ totally ordered for all $w \in W$. In a sense, here we are discussing 'upside-down trees', but we will refer to them as trees for convenience.

4.1 Poset Construction

Let W be an n-dimensional tree. Our aim is to build a copy of W using 'atomic' posets of the form

$$V = \{v_1, v_2\},$$
$$v_i \leq v_j \iff i \leq j.$$

Since our ring and homomorphism constructions are inductive procedures, it is useful to be able to reduce/increase the dimension of W at will. Let w_1 be the greatest element of W. We define the ith layer of W, denoted L_i, to be the set

$$L_i = \{w \in W : \text{ the maximum chain from } w \text{ to } w_1 \text{ is of length } i\}.$$

Lemma 4.2. *An n dimensional tree W can be partitioned into $n+1$ non-empty layers, that is for each $w \in W$ there exists a unique $i \in \{0, \ldots, n\}$ such that $w \in L_i$.*

Proof. It follows from the definition that the layers of W are disjoint. Since W is n-dimensional, the longest chain in W, $w_0 < \cdots < w_n$, is of length n. Since

$w_0^{\uparrow}$ is totally ordered, each chain starting at w_i and terminating at w_n is a 'sub-chain' of this chain, so the maximum chain beginning at w_i and terminating at w_n has length $n - i$. Hence $w_i \in L_{n-i}$ for $i = 0, \dots, n$, so each layer is non-empty. □

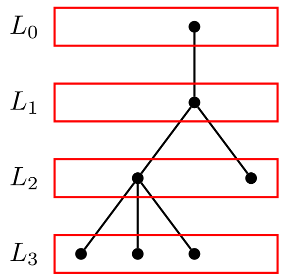


Figure 2: A 3-dimensional tree, partitioned into layers.

Lemma 4.3. *Each layer of a tree is 0-dimensional.*

Proof. Suppose L_i contains a chain of length 1. Then L_i contains elements $w_3 < w_2$. Let w_1 be the greatest element of W. The maximum chain in W starting at w_2 and terminating at w_1, $w_2 < \cdots < w_1$, is of length i, but the chain $w_3 < w_2 < \cdots < w_1$ is of length $i + 1$, which is a contradiction as $w_3 \in L_i$. □

Lemma 4.4. *Let W be an n-dimensional tree. Then $W \setminus L_n$ is an $n - 1$ dimensional tree.*

Proof. If w_1 is the greatest element of W then it is the greatest element of $W \setminus L_n$. Then the poset $W \setminus L_n$ cannot be disconnected as it has a greatest element. The layer L_{n-1} is a non-empty subset of $W \setminus L_n$, so W contains a chain of length $n-1$. Suppose $W \setminus L_n$ contains a chain $w_2 > w_3 > \cdots > w_{n+2}$ of length n. Then $w_1 \leq w_2 > w_3 > \cdots > w_{n+2}$ is a chain of length greater than n, so the chain contains an element of L_k for $k \geq n$, which is a contradiction. □

Lemma 4.5. *Let W be an n-dimensional tree with greatest element w_1. Then for all $w_2 \in W \setminus \{w_1\}$ there exists a unique $w_3 \in W$ which covers w_2. Furthermore, if $w_2 \in L_k$ then $w_3 \in L_{k-1}$.*

Proof. Since $w_2 \in L_k$ there is a chain of length k from w_2 to w_1, and since W is finite, one of these elements must cover w_2. Suppose $w_3 \neq w_4$ both cover w_2. Then $w_2 < w_3$ and $w_2 < w_4$, but $w_2^{\uparrow}$ is totally ordered so either $w_2 < w_3 < w_4$ or $w_2 < w_4 < w_3$, which means one of w_3 or w_4 does not cover w_2. Now suppose w_3 covers w_2 and $w_3 \notin L_{k-1}$. Since $w_3 > w_2$ we must have $w_3 \in L_l$ for $l < k-1$. Then the maximum chain from w_3 to w_1 is of length $l < k-1$, but the maximum chain from w_2 to w_1 is of length k. Since $w_2^{\uparrow}$ is totally ordered and all elements of the chain are contained in $w_2^{\uparrow}$, w_3 must be an element of the chain. But w_3

covers w_2, so there is a chain from w_3 to w_1 of length $k-1 > l$, which is a contradiction. $\square$

Theorem 4.6. *Let W be an n-dimensional tree. Define the posets*

$$\begin{aligned} X &= W \setminus L_n, \\ Y &= \left\{ v_{1,1}, v_{1,2}, \ldots, v_{|L_n|,1}, v_{|L_n|,2} \right\}, \\ Z &= \left\{ v_{i,2} \in Y \right\}, \end{aligned}$$

where the order relation on Y is

$$v_{i,j} \leq v_{k,l} \iff i = k \text{ and } j \leq l.$$

Let

$$\begin{gathered} f : Z \to X, \\ f(v_{i,2}) = w_j \iff w_j \text{ covers } w_{|W \setminus L_n| + i}, \end{gathered}$$

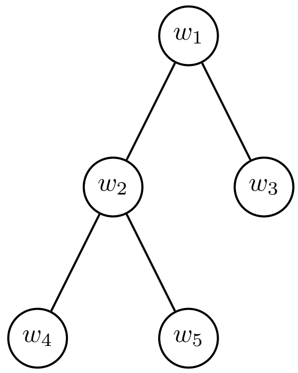
and let $g : Z \to Y$ be the inclusion function. Then $W \cong X \sqcup_Z Y$.

Proof. We claim that the function

$$\begin{gathered} \omega : W \to X \sqcup_Z Y, \\ \omega(w_i) = \begin{cases} w_i & \text{if } w_i \in W \setminus L_n, \\ v_{i-|W \setminus L_n|,1} & \text{if } w_i \in L_n, \end{cases} \end{gathered}$$

is an order isomorphism. We have $W \setminus L_n = X$ and $|Y \setminus Z| = |L_n|$, so ω is a surjection between two sets of the same size and is therefore bijective. Let $w_i \leq w_j$. If $w_i, w_j \in W \setminus L_n$ then $\omega(w_i) \leq \omega(w_j)$. If $w_i, w_j \in L_n$ then $w_i = w_j$ as L_n is 0-dimensional, so $\omega(w_i) = \omega(w_j)$.
Now suppose $\omega(w_i) \leq \omega(w_j)$. If $\omega(w_i), \omega(w_j) \in X$ then $w_i = \omega(w_i) \leq \omega(w_j) = w_j$. If $\omega(w_i), \omega(w_j) \in Y \setminus Z$ then $\omega(w_i) = \omega(w_j)$ as $Y \setminus Z$ is 0-dimensional, so $w_i = w_j$. If $\omega(w_i) \in Y \setminus Z, \omega(w_j) \in X$ then there exists $v_{k,l} \in Z$ such that $\omega(w_i) \leq g(v_{k,l})$ and $f(v_{k,l}) \leq \omega(w_j) = w_j$. Since g is the inclusion and $Y \setminus Z$ is 0-dimensional we have $\omega(w_i) = v_{k,l}$. Hence $v_{k,l} = v_{i-|W \setminus L_n|,1}$. Then $f(v_{i-|W \setminus L_n|,1}) = w_k$ implies w_k covers w_i in W, so $w_i \leq w_k \leq w_j$. $\square$

Example 4.7. Let W be the poset depicted below.



W

We first build the 1-dimensional tree $W \setminus L_2$. We define the posets

$$X_1 = \{v\}\,,$$
$$Y_1 = \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}\}\,,$$
$$Z_1 = \{v_{1,2}, v_{2,2}\}\,,$$

and define the function f_1 such that $f(v_{1,2}) = f(v_{2,2}) = v$. Then $W \setminus L_2 \cong X_1 \sqcup_{Z_1} Y_1$, so let $X_2 = X_1 \sqcup_{Z_1} Y_1$. Now define the posets

$$Y_2 = \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}\}\,,$$
$$Z_2 = \{v_{1,2}, v_{2,2}\}\,,$$

and the function f_2 such that $f(v_{1,2}) = f(v_{2,2}) = v_{1,2}$ (in X_2). Then $W \cong X_2 \sqcup_{Z_2} Y_2 = (X_1 \sqcup_{Z_1} Y_1) \sqcup_{Z_2} Y_2$.

4.2 Ring Construction

In this section we give an inductive procedure for constructing a ring with spectrum isomorphic to a given tree. The first three results give a base case for this procedure and the final three form the inductive step. The process is analagous to the poset construction procedure, and we shall construct the rings 'layer by layer'.

Base Case

Lemma 4.8. *Let W be a 1-dimensional tree with t elements. Then if X, Y, Z are as defined in this section's poset construction and we define the rings*

$$A = R_1 := k,$$
$$B = R_2 \times \cdots \times R_t := \left(k[X_1]_{\langle X_1 \rangle}\right)^{t-1},$$
$$C = R_2 / \langle X_1 \rangle \times \cdots \times R_t / \langle X_1 \rangle = \left(k[X_1]_{\langle X_1 \rangle} / \langle X_1 \rangle\right)^{t-1},$$

we have $X \cong \mathsf{Spec}\, A, Y \cong \mathsf{Spec}\, B$ and $Z \cong \mathsf{Spec}\, C$.

Proof. We have that $X \cong \mathsf{Spec}\, A$ as both are one element posets, so any function between them is an order isomorphism. In particular, the function

$$\alpha : X \to \mathsf{Spec}\, A,$$
$$\alpha(w_1) = \langle 0 \rangle,$$

is an order isomorphism.

Let $V_i = \{v_{i,j} \in Y\} = \{v_{i,1}, v_{i,2}\}$. Then $Y = V_1 \sqcup \cdots \sqcup V_{t-1}$, and $V_i \cong \mathsf{Spec}\, R_{i+1}$ for $i \in \{1, \ldots, t-1\}$. Hence $Y \cong \mathsf{Spec}\, R_2 \times \cdots \times R_t = \mathsf{Spec}\, B$. In particular, the function

$$\beta : Y \to \mathsf{Spec}\, B,$$
$$\beta(v_{i,j}) = \begin{cases} p_{R_{1+i}}^{-1}(\langle 0 \rangle) & \text{if } j = 1, \\ p_{R_{1+i}}^{-1}(\langle X_1 \rangle) & \text{if } j = 2, \end{cases}$$

is an order isomorphism.

We have that $Z \cong \mathsf{Spec}\, C$ as Z is a 0-dimensional poset with $t-1$ elements and C is the product of $t-1$ fields. Thus any bijective function between them is an order isomorphism. In particular, the function

$$\gamma : Z \to \mathsf{Spec}\, C,$$
$$\gamma(v_{i,2}) = \langle 1 \rangle \times \cdots \times \overbrace{\langle 0 \rangle}^{i\text{th place}} \times \cdots \times \langle 1 \rangle$$

is an order isomorphism. □

We stress that, like in Theorem 2.6, we can add throw-away indeterminates to our above rings without affecting any of the subsequent results, that is, for some $1 < p \leq q$ we can take $A = k(X_p, \dots, X_q)$ and $R_i = k[X_1, X_p, \dots, X_q]_{\langle X_1 \rangle}$ for $i \in \{2, \dots, t\}$, and all of the results in this section can be proven in a near-identical manner.

Lemma 4.9. *Let W be a 1-dimensional tree with t elements. Then, if X, Y, Z, f, g are as defined in this section's poset construction, A, B, C are as defined in Lemma 4.8 and we define*

$$\begin{gathered}\phi : A \to C,\\ \phi(f_1) = (f_1 + \langle X_1 \rangle, \dots, f_1 + \langle X_1 \rangle),\\ \psi : B \to C,\\ \psi(f_2, \dots, f_t) = (f_2 + \langle X_1 \rangle, \dots, f_t + \langle X_1 \rangle),\end{gathered}$$

we have that ψ is surjective and $\mathsf{Spec}\,\phi$ and $\mathsf{Spec}\,\psi$ are isomorphic to f and g respectively.

Proof. Recall that $\alpha : X \to \mathsf{Spec}\,A, \beta : Y \to \mathsf{Spec}\,B$ and $\gamma : Z \to \mathsf{Spec}\,C$ as defined in Lemma 4.8 are order isomorphisms. We immediately obtain that f and $\mathsf{Spec}\,\phi$ are isomorphic, as the domains are isomorphic, the codomains are isomorphic and the codomains are of size 1.

The function ψ is surjective as given $(c_2, \dots, c_{t-1}) \in C$ we can simply pick a representative f_i of c_i and we have $\psi(f_2, \dots, f_{t-1}) = (c_2, \dots, c_{t-1})$. To show that g and $\mathsf{Spec}\,\psi$ are isomorphic, we can show that $\beta \circ g(v_{i,2}) = \psi^{-1} \circ \gamma(v_{i,2})$. We have that $g(v_{i,2}) = v_{i,2}$, and that $\beta(v_{i,2}) = p^{-1}_{R_{1+i}}(\langle X_1 \rangle)$. We also have that

$$\gamma(v_{i,2}) = \mathfrak{p}_i := \langle 1 \rangle \times \dots \times \overbrace{\langle 0 \rangle}^{i\text{th place}} \times \dots \times \langle 1 \rangle,$$

so we must show that $p^{-1}_{R_{1+i}}(\langle X_1 \rangle) = \psi^{-1}(\mathfrak{p}_i)$. To begin with, note that

$$\begin{gathered}(1, \dots, \overbrace{0}^{j\text{th place}}, \dots, 1) \in p^{-1}_{R_{1+j}}(\langle 0 \rangle), p^{-1}_{R_{1+j}}(\langle X_1 \rangle) \text{ for } j \neq i, \text{ but}\\ \psi(1, \dots, \overbrace{0}^{j\text{th place}}, \dots, 1) = (1 + \langle X_1 \rangle, \dots, \overbrace{0 + \langle X_1 \rangle}^{j\text{th place}}, \dots, 1 + \langle X_1 \rangle) \notin \mathfrak{p}_i.\end{gathered}$$

Thus the only remaining possibilities are $p^{-1}_{R_{1+i}}(\langle 0 \rangle) = \psi^{-1}(\mathfrak{p}_i)$ and $p^{-1}_{R_{1+i}}(\langle X_1 \rangle) = \psi^{-1}(\mathfrak{p}_i)$. Now note that

$$\begin{gathered}\psi(0, \dots, \overbrace{X_1}^{i\text{th place}}, \dots, 0) = (0 + \langle X_1 \rangle, \dots, 0 + \langle X_1 \rangle) \in \mathfrak{p}_i, \text{ but}\\ (0, \dots, \overbrace{X_1}^{i\text{th place}}, \dots, 0) \notin p^{-1}_{R_{1+i}}(\langle 0 \rangle).\end{gathered}$$

Hence the only remaining possibility is $p^{-1}_{R_{1+i}}(\langle X_1 \rangle) = \psi^{-1}(\mathfrak{p}_i)$. Therefore $\mathsf{Spec}\,\psi$ and g are isomorphic. □

Proposition 4.10. *Let W be a 1-dimensional tree with t elements. Then there exists a ring D such that*

- *$D \subseteq R_1 \times \cdots \times R_t$ (that is, we have t homomorphisms $p_j : D \to R_j$, each of which is a composition of the inclusion $i : D \to R_1 \times \cdots \times R_t$ and the projection $p_{R_j} : R_1 \times \cdots \times R_t \to R_j$);*
- *the function*

$$\delta : W \to \mathsf{Spec}\, D,$$
$$\delta(w_j) = p_j^{-1}(\langle 0 \rangle)$$

is an order-isomorphism;

- *we have*

$$R_j = \begin{cases} k & \text{if } j = 1, \\ k[X_1]_{\langle X_1 \rangle} & \text{if } j \in \{2, \dots, t\}. \end{cases}$$

Proof. Let $D = A \times_C B$, where A, B and C are as defined in Lemma 4.8 and ϕ, ψ are as defined in Lemma 4.9. Then we have $D = A \times_C B \subseteq A \times B = R_1 \times \cdots \times R_t$, proving the property in the first bullet point.

Recall from this section's poset construction that the function

$$\omega : W \to X \sqcup_Z Y,$$
$$\omega(w_i) = \begin{cases} w_i & \text{if } w_i \in W \setminus L_n, \\ v_{i - |W \setminus L_n|} & \text{if } w_i \in L_n \end{cases}$$

is an order isomorphism. Then by Fontana's theorem, the function

$$\chi : X \sqcup_Z Y \to \mathsf{Spec}\, A \times_C B,$$
$$\chi(w) = \begin{cases} p_A^{-1} \circ \alpha(w) & \text{if } w \in X, \\ p_B^{-1} \circ \beta(w) & \text{if } w \in Y \setminus Z, \end{cases}$$

is an order isomorphism. We claim that $\delta = \chi \circ \omega$ is of the required form. Note that $\alpha(w) = p_A^{-1}(\langle 0 \rangle)$, and since $A = R_1$ we have $p_A = p_1$. We have that

$$Y \setminus Z = \{v_{i,1} \in Y : 1 \leq i \leq t-1\}.$$

Hence the restriction of β to $Y \setminus Z$ is given by $\beta(w) = p_{R_{1+i}}^{-1}(\langle 0 \rangle)$. Since we have that $p_i = p_{R_i} \circ p_B$ for $i \in \{2, \dots, t\}$, it follows that $\delta = \chi \circ \omega$ is of the required form, proving the property in the second bullet point.

The third bullet point follows directly from our definitions of $R_1, \dots, R_t$. $\square$

Inductive Step

We now generalise the previous results to trees of any dimension.

Lemma 4.11. *Let W be an n-dimensional tree. Then, if X, Y, Z are as defined in this section's poset construction and we define the rings*

$$B = R_{|W\setminus L_n|+1} \times \cdots \times R_{|W\setminus L_n|+|L_n|} := \left(k[X_1,\ldots,X_n]_{\langle X_n\rangle}\right)^{|L_n|},$$

$$C = R_{|W\setminus L_n|+1}/\left\langle X_n\right\rangle \times \cdots \times R_{|W\setminus L_n|+|L_n|}/\left\langle X_n\right\rangle = \left(k[X_1,\ldots,X_n]_{\langle X_n\rangle}/\left\langle X_n\right\rangle\right)^{|L_n|},$$

we have $Y \cong \mathsf{Spec}\, B$ and $Z \cong \mathsf{Spec}\, C$.

Proof. The posets Y and $\mathsf{Spec}\, B$ are isomorphic by the same reasoning as in Lemma 4.8. In particular, the function

$$\beta : Y \to \mathsf{Spec}\, B,$$

$$\beta(v_{i,j}) = \begin{cases} p^{-1}_{R_{|W\setminus L_n|+i}}(\langle 0\rangle) & \text{if } j = 1, \\ p^{-1}_{R_{|W\setminus L_n|+i}}(\langle X_n\rangle) & \text{if } j = 2, \end{cases}$$

is an order isomorphism.

Since C is the product of $|L_n|$ fields and Z is a 0-dimensional poset with $|L_n|$ elements, any bijective function between them is an order isomorphism. In particular, the function

$$\gamma : Z \to \mathsf{Spec}\, C,$$

$$\gamma(v_{i,2}) = \mathfrak{p}_i := \langle 1\rangle \times \cdots \times \overbrace{\langle 0\rangle}^{i\text{th place}} \times \cdots \times \langle 1\rangle$$

is an order isomorphism. □

Lemma 4.12. *Let W be an n-dimensional tree. Let X, Y, Z, f, g be as defined in this section's poset construction, B, C as defined in Lemma 4.11 and suppose there exists a ring A such that*

- *$A \subseteq R_1 \times \cdots \times R_{|W\setminus L_n|}$ (that is, we have $|W \setminus L_n|$ homomorphisms $p_j : A \to R_j$, each of which is a composition of the inclusion $i : A \to R_1 \times \cdots \times R_{|W\setminus L_n|}$ and the projection $p_{R_j} : R_1 \times \cdots \times R_{|W\setminus L_n|} \to R_j$);*
- *the function*

$$\alpha : W \setminus L_n \to \mathsf{Spec}\, A,$$
$$\alpha(w_j) = p_j^{-1}(\langle 0\rangle)$$

is an order isomorphism;

- *we have*

$$R_j = \begin{cases} k & \text{if } w_j \in L_0, \\ k[X_1,\ldots,X_l]_{\langle X_l\rangle} & \text{if } w_j \in L_l. \end{cases}$$

Then there exist homomorphisms ϕ and ψ such that $\mathsf{Spec}\,\phi$ *and* $\mathsf{Spec}\,\psi$ *are isomorphic to f and g respectively.*

Proof. Define the function

$$\sigma : \{1, \ldots, |L_n|\} \to \{1, \ldots, |W|\}\,,$$
$$\sigma(i) = j \iff f(v_{i,2}) = w_j.$$

Then we claim that the homomorphism

$$\phi : A \to C,$$
$$\phi(f_1, \ldots, f_{|W \setminus L_n|}) = \phi(r) = (p_{\sigma(1)}(r) + \langle X_n \rangle\,, \ldots, p_{\sigma(|L_n|)}(r) + \langle X_n \rangle)$$

is such that $\mathsf{Spec}\,\phi$ is isomorphic to f. In other words, we must show that $\alpha \circ f(v_{i,2}) = \phi^{-1} \circ \gamma(v_{i,2})$. Let w_j be the element of $W \setminus L_n$ that covers $w_{|W \setminus L_n|+i}$. Then $f(v_{i,2}) = w_j$ and $\sigma(i) = j$. We have that $\alpha(w_j) = p_j^{-1}(\langle 0 \rangle)$ and $\gamma(v_{i,2}) = \mathfrak{p}_i$, so we need to show that $\phi^{-1}(\mathfrak{p}_i) = p_j^{-1}(\langle 0 \rangle)$.

Suppose $r = (r_1, \ldots, r_{|W \setminus L|}) \in \phi^{-1}(\mathfrak{p})$. Then there exists $s = (s_1, \ldots, s_{|L_n|}) \in \mathfrak{p}_i$ such that $\phi(r) = s$. Since $s \in \mathfrak{p}_i$, we must have $s_i = 0$. Hence $p_{\sigma(i)}(r) + \langle X_n \rangle = p_j(r) + \langle X_n \rangle = r_j + \langle X_n \rangle$. This means $r_j \in \langle X_n \rangle$. But recall that $p_j : A \to R_j$. Since w_j is the element that covers w_i, we have that $w_j \in L_{n-1}$, so $R_j = k[X_1, \ldots, X_{n-1}, X_p \ldots, X_q]_{\langle X_l \rangle}$. The only element $r_j \in R_j$ such that $r_j + \langle X_n \rangle = 0 + \langle X_n \rangle$ is $r_j = 0$. Thus $r \in p_j^{-1}(\langle 0 \rangle)$, so $\phi^{-1}(\mathfrak{p}) \subseteq p_j^{-1}(\langle 0 \rangle)$. Now let $r = (r_1, \ldots, r_{|W \setminus L_n|}) \in p_j^{-1}(\langle 0 \rangle)$. Then $r_{\sigma(i)} = r_j = 0$, so $\phi(r) \in \mathfrak{p}_i$. Thus $r \in \phi^{-1}(\mathfrak{p}_i)$, so $p_j^{-1}(\langle 0 \rangle) = \phi^{-1}(\mathfrak{p}_i)$. Therefore $\mathsf{Spec}\,\phi$ is isomorphic to f.

It can be shown that the homomorphism

$$\psi : B \to C,$$
$$\psi(f_{|W \setminus L_n|+1}, \ldots, f_{|W \setminus L_n|+|L_n|}) = (f_{|W \setminus L_n|+1} + \langle X_n \rangle\,, \ldots, f_{|W \setminus L_n|+|L_n|} + \langle X_n \rangle)$$

is surjective and that $\mathsf{Spec}\,\psi$ is isomorphic to g by near-identical methods to those used in the proof of Lemma 4.9. ☐

Theorem 4.13. *Let W be an n-dimensional tree. Then there exists a ring D such that*

- $D \subseteq R_1 \times \cdots \times R_{|W|}$ *(that is, we have $|W|$ homomorphisms $p_j : D \to R_j$, each of which is a composition of the inclusion $i : D \to R_1 \times \cdots \times R_{|W|}$ and the projection $p_{R_j} : R_1 \times \cdots \times R_{|W|} \to R_j$);*
- *the function*

$$\delta : W \to \mathsf{Spec}\, D,$$
$$\delta(w_j) = p_j^{-1}(\langle 0 \rangle)$$

is an order isomorphism;

- *we have*

$$R_j = \begin{cases} k & \textit{if } w_j \in L_0, \\ k[X_1, \dots, X_l]_{\langle X_l \rangle} & \textit{if } w_j \in L_l. \end{cases}$$

Proof. We proceed by induction on the dimension of W. The proof of our base case follows from Proposition 4.10. We now assume the induction hypothesis for trees of dimension 1 to $n-1$. Now let W be a tree of dimension n. Then by Lemma 4.4 the poset $W \setminus L_n$ is a tree of dimension $n-1$, so by assumption there exists a ring A such that

- $A \subseteq R_1 \times \cdots \times R_{|W \setminus L_n|}$ (that is, we have $|W \setminus L_n|$ homomorphisms $p_j : A \to R_j$, each of which is a composition of the inclusion $i : A \to R_1 \times \cdots \times R_{|W \setminus L_n|}$ and the projection $p_{R_j} : R_1 \times \cdots \times R_{|W \setminus L_n|} \to R_j$);
- the function

$$\alpha : W \setminus L_n \to \mathsf{Spec}\, A,$$
$$\alpha(w_j) = p_j^{-1}(\langle 0 \rangle)$$

  is an order isomorphism;
- we have

$$R_j = \begin{cases} k & \text{if } w_j \in L_0, \\ k[X_1, \dots, X_l]_{\langle X_l \rangle} & \text{if } w_j \in L_l. \end{cases}$$

Let B, C be the rings constructed in Lemma 4.11 and ϕ, ψ be the homomorphisms constructed in Lemma 4.12. Let $D = A \times_C B$. We have that $A \times_C B \subseteq A \times B = R_1 \times \cdots \times R_{|W|}$, proving the property in the first bullet point, and the property in the third bullet point is satisfied by the definitions of $R_1, \dots, R_{|W|}$.

The isomorphism obtained from this section's poset construction is

$$\omega : W \to X \sqcup_Z Y,$$
$$\omega(w_i) = \begin{cases} w_i & \text{if } w_i \in W \setminus L_n, \\ v_{i-|W \setminus L_n|} & \text{if } w_i \in L_n, \end{cases}$$

and the isomorphism obtained from Fontana's Theorem is

$$\chi : X \sqcup_Z Y \to \mathsf{Spec}\, A \times_C B,$$
$$\chi(w) = \begin{cases} p_A^{-1} \circ \alpha(w) & \text{if } w \in X, \\ p_B^{-1} \circ \beta(w) & \text{if } w \in Y \setminus Z. \end{cases}$$

We claim that $\delta = \chi \circ \omega$ satisfies the required properties. We have that $\alpha(w_i) = p_{R_i}^{-1}(\langle 0 \rangle)$ for $w_i \in |W \setminus L_n|$ and $p_i = p_{R_i} \circ p_A$. For similar reasons as in Proposition 4.10, we also have that $\beta(v_{i,1}) = p_{|W \setminus L_n|+i}^{-1}(\langle 0 \rangle)$. It then follows that δ is of the required form. $\square$

Again, we stress that in this final ring construction, the addition of the throw-away indeterminates $X_p, \ldots, X_q$ does not affect the final result. Indeed, it is only slightly more work to verify that, given $n < p \leq q$, we can find a ring which satisfies the first two bullet points, but we instead have

$$R_j = \begin{cases} k(X_p, \ldots X_q) & \text{if } w_j \in L_0, \\ k[X_1, \ldots, X_l, X_p, \ldots, X_q] & \text{if } w_j \in L_l. \end{cases}$$

Example 4.14. Let W be as shown in Example 4.7. Let $R_1 = k, R_2 = R_3 = k[X_1]_{\langle X_1 \rangle}$ and $R_4 = R_5 = k[X_1, X_2]_{\langle X_2 \rangle}$. Let $A_1 = k, B_1 = R_2 \times R_3$ and $C_1 = R_2/\langle X_1 \rangle \times R_3/\langle X_1 \rangle$ and let

$$\begin{gathered} \phi : A_1 \to C_1, \\ \phi(f_1) = (f_1 + \langle X_1 \rangle, f_1 + \langle X_1 \rangle), \\ \psi : B_1 \to C_1, \\ \psi(f_2, f_3) = (f_2 + \langle X_1 \rangle, f_3 + \langle X_1 \rangle). \end{gathered}$$

Then let $A_2 = A_1 \times_{C_1} B_1$, $B_2 = R_4 \times R_5$ and $C_2 = R_4/\langle X_2 \rangle \times R_5/\langle X_2 \rangle$. We have $\sigma(1) = \sigma(2) = 2$, so define

$$\begin{gathered} \phi : A_2 \to C_2, \\ \phi(f_1, f_2, f_3) = (f_2 + \langle X_2 \rangle, f_2 + \langle X_2 \rangle), \\ \psi : B_2 \to C_2, \\ \psi(f_4, f_5) = (f_4 + \langle X_2 \rangle, f_5 + \langle X_2 \rangle). \end{gathered}$$

Then $D = A_2 \times_{C_2} B_2 = (A_1 \times_{C_1} B_1) \times_{C_2} B_2$ is such that $W \cong \mathsf{Spec}\, D$.

4.3 Homomorphism Construction

Similar to the previous section, we will require for this homomorphism construction that an order preserving function F satisfies an extra restriction. We begin our homomorphism construction by defining the criterion which a 'layer-compressing' function must fulfil.

Definition 4.15 (Layer-Compressing Function)**.** Let $W, \tilde{W}$ be trees and $F : W \to \tilde{W}$ an order-preserving function. We say that F is layer-compressing if is satisfies the following property:

$$w_1 \leq w_2 \text{ with } w_1 \in L_i, w_2 \in L_j \implies F(w_1) \in \tilde{L}_k, F(w_2) \in \tilde{L}_l \text{ where } l-k \leq j-i.$$

Lemma 4.16. *Let $F : W \to \tilde{W}$ be a layer-compressing function. If w_1 is covered by w_2 in W then either*

- $F(w_1) = F(w_2)$*, or*
- $F(w_1)$ *is covered by* $F(w_2)$*.*

Proof. By Lemma 4.5, if $w_2 \in L_i$ then $w_1 \in L_{i+1}$. Since F is a layer-compressing function, if $F(w_2) \in \tilde{L}_j$ then $F(w_1) \in \tilde{L}_k$, where $1 = i + 1 - i \geq k - j$, so $k = j$ or $k = j + 1$. If $k = j$ then $F(w_1), F(w_2) \in \tilde{L}_j$ which is a 0-dimensional poset. But $F(w_1) \leq F(w_2)$ as F is order-preserving, so $F(w_1) = F(w_2)$. Suppose $k = j+1$. Then there exists a unique element $w_3 \in \tilde{L}_j$ that covers $F(w_1) \in \tilde{L}_{j+1}$. Then $F(w_2) \geq w_3 > F(w_1)$, but $F(w_2), w_3 \in \tilde{L}_j$ which is 0-dimensional, so $F(w_2) = w_3$ and hence covers w_1. □

Harking back to our second 1-dimensional homomorphism construction, we will track the behaviour of subsets of the tree, build homomorphisms which correspond to them, and show that the 'fibre product constraint' ($\phi(a) = \psi(b)$) is satisfied. Building the homomorphisms is simple, as our ring construction guarantees that we have an individual ring corresponding to each element of the tree. However it is cumbersome to show that the fibre product constraint is satisfied directly, so we prove the following result.

Proposition 4.17 (Equivalence of the Fibre Product Constraint)**.** *Let W be an n-dimensional tree with D as constructed in Theorem 4.13. Then $r = \left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W|}}{g_{|W|}}\right) \in D$ if and only if*

$$\frac{f_i(X_1, \ldots, X_k, 0)}{g_i(X_1, \ldots, X_k, 0)} = \frac{f_j(X_1, \ldots, X_k)}{g_j(X_1, \ldots, X_k)}$$

for all $w_i \in L_{k+1}, w_j \in L_k$ such that w_j covers w_i.

Proof. We proceed by induction on the dimension of W. Let W be a 1-dimensional tree. First let $r = \left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W|}}{g_{|W|}}\right) \in D$. Then we have

$$\phi\left(\frac{f_1}{g_1}\right) = \psi\left(\frac{f_2}{g_2}, \ldots, \frac{f_{|W|}}{g_{|W|}}\right).$$

By definition of ϕ and ψ, we have $\frac{f_1}{g_1} + \langle X_1 \rangle = \frac{f_i(X_1)}{g_i(X_1)} + \langle X_1 \rangle$ for all $i \in \{2, \ldots, |W|\}$. Thus there exists $\frac{f(X_1)}{g(X_1)} \in k[X_1]_{\langle X_1 \rangle}$ such that

$$\frac{f_1}{g_1} = \frac{f_i(X_1)}{g_i(X_1)} + X_1 \frac{f(X_1)}{g(X_1)}.$$

Since multiples of X_1 are non-units in $k[X_1]_{\langle X_1 \rangle}$, it is valid to evaluate $\frac{f_i(X_1)}{g_i(X_1)} + X_1 \frac{f(X_1)}{g(X_1)}$ at $X_1 = 0$, and we obtain

$$\frac{f_1}{g_1} = \frac{f_i(0)}{g_i(0)}.$$

Now assume $\frac{r_1}{s_1} := \frac{f_1}{g_1} = \frac{f_i(0)}{g_i(0)}$. Our base case is complete if we can find a solution $\frac{f}{g} \in k[X_1]_{\langle X_1 \rangle}$ of the equation

$$\begin{aligned} \frac{f_i(X_1)}{g_i(X_1)} &= \frac{r_1}{s_1} + X_1 \frac{f(X_1)}{g(X_1)} \\ &= \frac{r_1 g(X_1) + s_1 X_1 f(X_1)}{s_1 g(X_1)}. \end{aligned}$$

We have that

$$\frac{f(X_1)}{g(X_1)} = \frac{s_1 X_1^{-1}(f_i(X_1) - r_1 s_1^{-1} g_i(X_1)}{s_1^{-1} g_i(X_1)}$$

is a solution, but the solution is only valid if $f_i(X_1) - r_0 s_0^{-1} g_i(X_1) \in \langle X_1 \rangle$ i.e. if the constant term is zero. By assumption, we have

$$f_i(0) - r_0 s_0^{-1} g_i(0) = r_0 - r_0 s_0^{-1} s_0 = 0,$$

proving that the solution is valid and concluding the base case.

Now assume that the induction hypothesis holds for trees of dimensions 1 to $n-1$. Let W be an n-dimensional tree. First let $r = \left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W|}}{g_{|W|}} \right) \in D \subseteq A \times_C B$. This means $\left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W \setminus L_n|}}{g_{|W \setminus L_n|}} \right) \in A$, and A is a ring constructed by applying Theorem 4.13 to $W \setminus L_n$, which by Lemma 4.4 is an $n-1$-dimensional tree. Therefore by assumption, the desired property holds for $k \in \{0, \ldots, n-2\}$, but it remains to be proven for $k = n-1$. If $r \in D$ then we have

$$\phi \left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W \setminus L_n|}}{g_{|W \setminus L_n|}} \right) = \psi \left(\frac{f_{|W \setminus L_n|+1}}{g_{|W \setminus L_n|+1}}, \ldots, \frac{f_{|W|}}{g_{|W|}} \right).$$

Hence if $w_i \in L_n$ is covered by $w_j \in L_{n-1}$ then

$$\frac{f_i(X_1, \ldots, X_n)}{g_i(X_1, \ldots, X_n)} + \langle X_n \rangle = \frac{f_j(X_1, \ldots, X_{n-1})}{g_j(X_1, \ldots, X_{n-1})} + \langle X_n \rangle .$$

In other words, there exists $\frac{f}{g} \in k[X_1, \ldots, X_n]_{\langle X_n \rangle}$ such that

$$\frac{f_i(X_1,\ldots,X_n)}{g_i(X_1,\ldots,X_n)} = \frac{f_j(X_1,\ldots,X_{n-1})}{g_j(X_1,\ldots,X_{n-1})} + X_n \frac{f(X_1,\ldots,X_n)}{g(X_1,\ldots,X_n)}.$$

Since multiples of X_n are non-units in $k[X_1, \ldots, X_n]_{\langle X_n \rangle}$, it is valid to evaluate the function at $X_n = 0$, hence we have

$$\frac{f_i(X_1,\ldots,X_{n-1},0)}{g_i(X_1,\ldots,X_{n-1},0)} = \frac{f_j(X_1,\ldots,X_{n-1})}{g_j(X_1,\ldots,X_{n-1})}.$$

Now assume we have $\frac{f_i(X_1,\ldots,X_{n-1},0)}{g_i(X_1,\ldots,X_{n-1},0)} = \frac{f_j(X_1,\ldots,X_{n-1})}{g_j(X_1,\ldots,X_{n-1})}$. The induction step is complete if we can find a solution $\frac{f}{g} \in k[X_1, \ldots, X_n]_{\langle X_n \rangle}$ to the equation

$$\frac{f_i(X_1,\ldots,X_n)}{g_i(X_1,\ldots,X_n)} = \frac{f_j(X_1,\ldots,X_{n-1})}{g_j(X_1,\ldots,X_{n-1})} + X_n \frac{f(X_1,\ldots,X_n)}{g(X_1,\ldots,X_n)} \quad = \frac{f_j g + g_j X_n f}{g_j g}.$$

Recalling that f_j, g_j are both units in $k[X_1, \ldots, X_n]_{\langle X_n \rangle}$, we have that

$$\frac{f(X_1,\ldots,X_n)}{g(X_1,\ldots,X_n)} = \frac{g_j X_n^{-1}(f_i(X_1,\ldots,X_n) - f_j g_j^{-1} g_i(X_1,\ldots,X_n))}{g_j^{-1} g(X_1,\ldots,X_n)}$$

is a solution, but the solution is only valid if

$$(f_i(X_1,\ldots,X_n) - f_j g_j^{-1} g_i(X_1,\ldots,X_n)) \in \langle X_n \rangle$$

i.e. $(f_i(X_1,\ldots,X_{n-1},0) - f_j g_j^{-1} g_i(X_1,\ldots,X_{n-1},0)) = 0$, which is true by assumption. □

If we add more units to the ring, as we will for the homomorphism construction, the proof is near-indentical but the statement of the proposition changes to state that $\left(\frac{f_1}{g_1}, \ldots, \frac{f_{|W|}}{g_{|W|}}\right) \in D$ if and only if

$$\frac{f_i(X_1,\ldots,X_k,0,X_p,\ldots,X_q)}{g_i(X_1,\ldots,X_k,0,X_p,\ldots,X_q)} = \frac{f_j(X_1,\ldots,X_k,X_p,\ldots,X_q))}{g_j(X_1,\ldots,X_k,X_p,\ldots,X_q))}.$$

Theorem 4.18. *Let $W, \tilde{W}$ be trees and $F : W \to \tilde{W}$ a layer-compressing function. Then there exist rings $D, \tilde{D}$ and a homomorphism $\Phi : \tilde{D} \to D$ such that $W \cong \mathsf{Spec}\, D, \tilde{W} \cong \mathsf{Spec}\, \tilde{D}$ and $\mathsf{Spec}\, \Phi$ is isomorphic to F.*

Proof. Suppose W is of dimension n and $\tilde{W}$ is of dimension $\tilde{n}$. Let $D, \tilde{D}$ be rings constructed using Theorem 4.13 using $W, \tilde{W}$ respectively, but for simplicity of notation we let indeterminates in the ring $\tilde{D}$ be given as $X_1, \ldots, X_{\tilde{n}}$ and indeterminates in the ring D be given as $Y_1, \ldots, Y_{n+\tilde{n}}$. For convenience, let $\tilde{D}$ be the ring described at the end of the chapter with no extra throw-away indeterminates, and let D be the ring constructed by taking $p = n + 1$ and

$q = n + \tilde{n}$. Recall that the elements of $W, \tilde{W}$ are labelled $w_1, \dots, w_{|W|}$ and $\tilde{w}_1, \dots, \tilde{w}_{|\tilde{W}|}$ and define the following function:

$$\sigma : \{1, \dots, |W|\} \to \left\{1, \dots, \left|\tilde{W}\right|\right\},$$
$$\sigma(i) = j \iff F(w_i) = \tilde{w}_j.$$

Suppose $F(w_1) \in L_m$. Define the following family $\{H_i\}_{i=1}^{\tilde{n}}$ of sets:

$$H_i = \begin{cases} \emptyset & \text{if } i \leq m \text{ or } i > m + n, \\ \{i - m\} & \text{if } m < i \leq m + n. \end{cases}$$

The sets H_i are pairwise disjoint, so we can use them to define a function η like that in Lemma 2.14. The function is

$$\eta : \{X_1, \dots, X_{\tilde{n}}\} \to \{Y_1, \dots, Y_n, Y_{n+1}, \dots, Y_{n+\tilde{n}}\},$$
$$\eta = \begin{cases} \prod_{j \in H_i} Y_j & \text{if } H_i \text{ is non-empty}, \\ Y_{n+i} & \text{otherwise}, \end{cases}$$

and by the same reasoning as in the proof of Lemma 2.14, this gives rise to an algebraically independent set $\{\eta(X_i)\}_{i=1}^{\tilde{n}}$. If $w_i \in L_k$ and $F(w_i) = \tilde{w}_{\sigma(i)} \in \tilde{L}_l$ then $\tilde{D}_{\sigma(i)} = k[X_1, \dots, X_l]_{\langle X_l \rangle}$. We want to check that the homomorphism

$$\Psi_i : k[X_1, \dots, X_l] \to R_i,$$
$$\Psi_i(f(X_1, \dots, X_l)) = f(\eta(X_1), \dots, \eta(X_l)),$$

is well-defined i.e. $\eta(X_i) \in \{Y_1, \dots, Y_k, Y_{n+1}, \dots, Y_{n+\tilde{n}}\}$ for $p \in \{1, \dots, l\}$. If $\eta(X_p) = Y_{n+p}$ then $\eta(X_p) \in \{Y_1, \dots, Y_k, Y_{n+1}, \dots, Y_{n+\tilde{n}}\}$ so assume $\eta(X_p) = Y_{p-m}$. Then we either need $p \leq m$, but $p > m$ by assumption, or $p - m > j$. Note $p \leq l$. Recall that $w_1 \in L_0, F(w_1) \in \tilde{L}_m$ and $w_i \in L_k$ with $F(w_i) = \tilde{w}_j \in \tilde{L}_l$. Then, since F is layer compressing, we have $k - 0 \geq l - m \geq p - m$, so we never have $p - m > k$. Hence Ψ_i is well-defined.

Let $S = k[X_1, \dots, X_l] \setminus \langle X_l \rangle$. We want to check that $g \in S$ implies $\Psi_i(g)$ is a unit of R_i. Suppose $g \in S$ but $\Psi_i(g)$ is a non-unit of R_i. Then $\Psi_i(g) \in \langle Y_k \rangle$, so $Y_k \mid \Psi_i(g)$. Then there exists some $\eta(X_p) = Y_k$, so we can write $g = g' X_p$. If we can show that $\eta(X_p) = Y_k$ implies $p = l$, then we are done. Note that $p \leq l$, so if $p \neq l$ then $p < l$. Then we have $p - m < l - m \leq k$, so if $\eta(X_p) = Y_k$ then $p = l$. Then $g' X_l \in \langle X_l \rangle$, so $g \notin S$, meaning $\Psi_i(g)$ is a unit of R_i for all $g \in S$. Thus by the universal property of localisation we have that

$$\Phi_i : \tilde{R}_{\sigma(i)} \to R_i,$$
$$\Phi_i \left(\frac{f}{g} \right) = \frac{\Psi_i(f)}{\Psi_i(g)},$$

is a well-defined homomorphism. Now define the homomorphism

$$\Phi : \tilde{D} \to D,$$

$$\Phi\left(\frac{f_1}{g_1}, \dots, \frac{f_{|\tilde{W}|}}{g_{|\tilde{W}|}}\right) = \left(\Phi_1\left(\frac{f_{\sigma(1)}}{g_{\sigma(1)}}\right), \dots, \Phi_{|W|}\left(\frac{f_{\sigma(|W|)}}{g_{\sigma(|W|)}}\right)\right).$$

We claim that Φ is well-defined (that is, elements in its image satisfy the fibre product constraint), and we use Proposition 4.17 to prove this claim. Suppose $w_i \in L_{k+1}$ is covered by $w_j \in L_k$ in W. Then by Lemma 4.16, we have either $\tilde{w}_{\sigma(i)} = F(w_i) = F(w_j) = \tilde{w}_{\sigma(j)}$ or that $\tilde{w}_{\sigma(i)} = F(w_i)$ is covered by $\tilde{w}_{\sigma(j)} = F(w_j)$. If $F(w_i) = F(w_j)$ then $\sigma(i) = \sigma(j)$, so $\Phi_i\left(\frac{f_{\sigma(i)}}{g_{\sigma(i)}}\right) = \Phi_j\left(\frac{f_{\sigma(j)}}{g_{\sigma(j)}}\right)$. Hence the evaluation of both functions at $Y_{k+1} = 0$ is equal, so the fibre product constraint is satisfied. If $\tilde{w}_{\sigma(i)} = F(w_i)$ is covered by $\tilde{w}_{\sigma(j)} = F(w_j)$ then we have that

$$\frac{f_{\sigma(i)}(X_1, \dots, X_l, 0)}{g_{\sigma(i)}(X_1, \dots, X_l, 0)} = \frac{f_{\sigma(j)}(X_1, \dots, X_l)}{g_{\sigma(j)}(X_1, \dots, X_l)}.$$

Under Φ_i and Φ_j we obtain

$$\Phi_i\left(\frac{f_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}{g_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}\right) = \frac{f_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), \eta(X_{l+1}))}{g_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), \eta(X_{l+1}))}$$
$$\Phi_j\left(\frac{f_{\sigma(j)}(X_1, \dots, X_l)}{g_{\sigma(j)}(X_1, \dots, X_l)}\right) = \frac{f_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l)}{g_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l))}.$$

We claim that $\eta(X_{l+1}) = Y_{k+1}$. An analogy to an earlier argument tells us that $\eta(X_p) = Y_{k+1}$ implies $p = l+1$, so it only remains to be shown that $m < l+1 \leq m+n$. We have $w_1 \in L_0$ with $F(w_1) \in \tilde{L}_m$ and $w_i \geq w_1$ with $F(w_i) \in \tilde{L}_l$, so $l \geq m$, meaning $l+1 > m$. We have that $k+1 \leq n$, so $n \geq k+1-0 \geq l+1-m$, meaning $l+1 \leq n+m$. Thus $\eta(X_{l+1}) = Y_{k+1}$. Then we have

$$\Phi_i\left(\frac{f_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}{g_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}\right) = \frac{f_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), Y_{k+1})}{g_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), Y_{k+1})}$$
$$\Phi_j\left(\frac{f_{\sigma(j)}(X_1, \dots, X_l)}{g_{\sigma(j)}(X_1, \dots, X_l)}\right) = \frac{f_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l)}{g_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l))},$$

implying

$$\begin{aligned}\Phi_i\left(\frac{f_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}{g_{\sigma(i)}(X_1, \dots, X_l, X_{l+1})}\right)\Bigg|_{Y_{k+1}=0} &\\ = \frac{f_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), 0)}{g_{\sigma(i)}(\eta(X_1), \dots, \eta(X_l), 0)} &\\ = \frac{f_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l)}{g_{\sigma(j)}(\eta(X_1), \dots, \eta(X_l))} &\\ = \Phi_j\left(\frac{f_{\sigma(j)}(X_1, \dots, X_l)}{g_{\sigma(j)}(X_1, \dots, X_l)}\right) &\end{aligned}$$

meaning the fibre product constraint is satisfied.

Finally we show that $\mathsf{Spec}\,\Phi$ is isomorphic to F. We have that

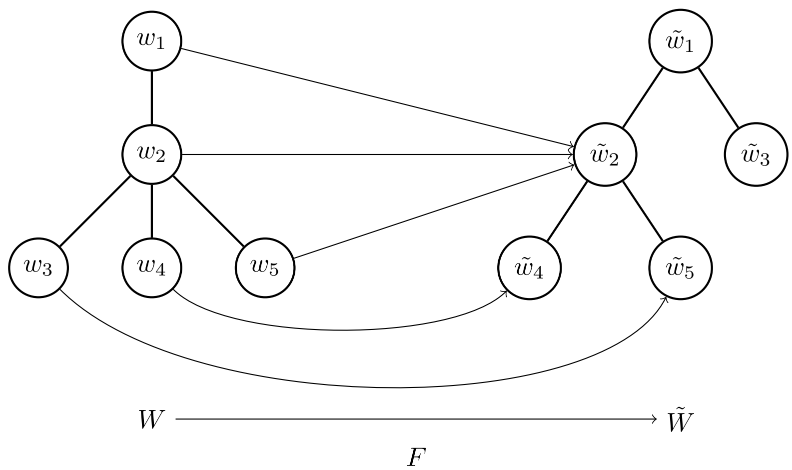
$$\begin{aligned} \delta : W &\to \mathsf{Spec}\,D, \\ \delta(w_j) &= p_j^{-1}(\langle 0 \rangle), \\ \tilde{\delta} : \tilde{W} &\to \mathsf{Spec}\,\tilde{D}, \\ \delta(\tilde{w}_j) &= \tilde{p}_j^{-1}(\langle 0 \rangle), \end{aligned}$$

are order isomorphisms. We must show that $\tilde{\delta} \circ F(w_i) = \Phi^{-1} \circ \delta(w_i)$, so it suffices to show that $\tilde{p}_{\sigma(i)}^{-1}(\langle 0 \rangle) = \Phi^{-1} \circ p_i^{-1}(\langle 0 \rangle)$ for all i. Using a result which can be proven in the same way as Lemma 3.19, we have that

$$\tilde{p}_{\sigma(i)}^{-1}(\langle 0 \rangle) = \tilde{p}_{\sigma(i)}^{-1} \circ \Phi_i^{-1}(\langle 0 \rangle) = \Phi^{-1} \circ p_i^{-1}(\langle 0 \rangle),$$

proving that $\mathsf{Spec}\,\Phi$ is isomorphic to F. □

Example 4.19. Let $F : W \to \tilde{W}$ be the layer-compressing function pictured below.



In Example 4.14 we found a ring $\tilde{D}$ such that $\tilde{W} \cong \mathsf{Spec}\, \tilde{D}$. If we let $R_1 = k(Y_3, Y_4), R_2 = k[Y_1, Y_3, Y_4]_{\langle Y_1 \rangle}, R_3 = R_4 = R_5 = k[Y_1, Y_2, Y_3, Y_4]_{\langle Y_2 \rangle}$ and let

$$\begin{aligned} A_1 &= R_1, \\ B_1 &= R_2, \\ C_1 &= R_2/\langle Y_1 \rangle, \\ A_2 &= A_1 \times_{C_1} B_1, \\ B_2 &= R_3 \times R_4 \times R_5, \\ C_2 &= R_3/\langle Y_2 \rangle \times R_4/\langle Y_2 \rangle \times R_5/\langle Y_2 \rangle, \end{aligned}$$

(where A_2 is the fibre product over ϕ, ψ as obtained in the ring construction), then $D = A_2 \times_{C_2} B_2$ is such that $W \cong \mathsf{Spec}\, D$. We have $\sigma(1) = \sigma(2) = \sigma(5) = 2, \sigma(3) = 5$ and $\sigma(4) = 5$. We have $m = 1$, so $H_1 = \emptyset, H_2 = \{1\}$. These sets induce the function

$$\begin{gathered} \eta : \{X_1, X_2\} \to \{Y_1, Y_2, Y_3, Y_4\}, \\ \eta(X_1) = Y_3, \quad \eta(X_2) = Y_1. \end{gathered}$$

Finally, the homomorphism

$$\Phi : \tilde{D} \to D,$$

$$\begin{aligned} &\Phi\left(\frac{f_1}{g_1}, \frac{f_2(X_1)}{g_2(X_1)}, \frac{f_3(X_1)}{g_3(X_1)}, \frac{f_4(X_1, X_2)}{g_4(X_1, X_2)}, \frac{f_5(X_1, X_2)}{g_5(X_1, X_2)}\right) \\ &\quad = \left(\frac{f_2(Y_3)}{g_2(Y_3)}, \frac{f_2(Y_3)}{g_2(Y_3)}, \frac{f_5(Y_3, Y_1)}{g_5(Y_3, Y_1)}, \frac{f_4(Y_3, Y_1)}{g_4(Y_3, Y_1)}, \frac{f_2(Y_3)}{g_2(Y_3)}\right), \end{aligned}$$

is such that $\mathsf{Spec}\, \Phi$ is isomorphic to F.

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