Matrix Algebras, Matrix Varieties and the Utility of Weyr Matrices

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Plagiarism Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

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Chapter 0

Introduction

Any student of linear algebra will be familiar with the Cayley-Hamilton theorem: every matrix satisfies its own characteristic polynomial. When applied to the matrix algebra $F[A]$, defined to be the linear span of all powers of $A \in M_n(F)$, this theorem tells us that only the matrices $I, A, A^2, \ldots, A^{n-1}$ are required to span $F[A]$. That is, $F[A]$ has dimension bounded above by n. One might therefore extrapolate that for $F[A, B]$ defined to be the linear span of all products of powers of $A, B \in M_n(F)$, the dimension of $F[A, B]$ can go as high as n^2. However, in [5] (1961), Gerstenhaber gave a proof of the counter-intuitive result that for commuting matrices A and B, the dimension of $F[A, B]$ will never exceed n. This result has come to be known as *Gerstenhaber's Theorem*, despite the fact that, as noted in [6], that the theorem is 'an immediate corollary to an old result of Motzkin and Taussky'.

Gerstenhaber's result integrates the fields of matrix theory and algebraic geometry, as his proof utilises the *irreducibility* of commuting pairs of $n \times n$ matrices, denoted $\mathcal{C}(2, n)$. In his paper, Gerstenhaber posed the question of whether similar sets of commuting k-tuples of $n \times n$ matrices, denoted $\mathcal{C}(k, n)$, are also irreducible. For most values of k and n this problem has been resolved, but some cases remain open as of 2022.

Several advances related to this problem have used the properties of the *Weyr canonical form*, a little-known canonical form dual to Jordan normal form. The original discovery of the Weyr form is credited to Czech mathematician Eduard Weyr, first appearing in the paper [27] (1885). The better-known Jordan normal form was, as alleged in [8], first discovered by Weierstrass in 1868 two years before Jordan's discovery of the form. According to [22, p95], Jordan normal form did not become the 'canonical matrix form of choice until the 1930s'. Regardless, awareness of the Weyr form has somewhat vanished: a sentiment perfectly illustrated by the number of times the canonical form has been rediscovered. The first chapter of this report provides an introduction to the Weyr canonical form, and discusses the properties it possesses which are advantageous for its use in commutative matrix algebra problems. The most crucial of these properties is the form of the *centraliser* of a Weyr matrix, that is, the deduction that matrices commuting with a Weyr matrix have a special *block upper-triangular* form. The first mention of this property appears to be in [23] (2005), where the Weyr form was reinvented as the 'H-form'. The first chapter will also cover the duality theorem relating the Jordan and Weyr forms of a matrix, as well as their characteristics. The permutation used to relate the canonical forms was known to Belitskii in [2] (first published in Russian in 1983), who re-derived the Weyr form as a 'modified normal form', though the duality of the structures associated to the canonical forms does not appear until Shapiro's paper [25] (1999) on the Weyr characteristic, known as the Weyr structure in this report and other literature. This paper is credited in [22, p81] with establishing the name of the canonical form used currently.

The second chapter compiles a number of results arising from problems on bounding the dimension of a matrix algebra. Whilst algebras generated by two commuting matrices have dimension bounded above my matrix size, algebras generated by four commuting matrices are known to violate this bound and even greater bounds according to Bergman's paper [3] (2013), whose construction will be examined in the second chapter. Whether all matrix algebras generated by a commuting triple of matrices have dimension bounded above by matrix size is an open problem known as *Gerstenhaber's problem*. Such commuting triples are believed to exist, and much justification for this belief is presented across the papers [9] and [24] through connections with analogous problems related to commutative rings, and inabilities for current methods of proof of Gerstenhaber's theorem to extend to a three-matrix analogue. Though no such commuting triples are known, a computing strategy which exploits the properties of Weyr matrices has been developed to search for them. The second chapter closes with an explanation of why Weyr matrices are desirable for this purpose, as well as an outline of the algorithm used, and an example of a triple generated and checked by the algorithm.

In the final chapter, we investigate the ties binding problems on the dimension of a matrix algebra and problems in algebraic geometry. Our knowledge of which matrix sizes n may admit triples generating an algebra of dimension exceeding n is connected with the reducibility of $\mathcal{C}(3, n)$. This set admits the structure of an *affine variety*, meaning it can be identified with a set of points at which some collection of polynomials vanishes. We develop the necessary results from algebraic geometry required for an assessment of these matrix variety problems. (Ir)reducibility of $\mathcal{C}(k,n)$ is established in the report for all but 24 cases, 18 of which are still unsolved. To do this we rely on only four theorems. The first is the result of Motzkin and Taussky mentioned earlier, which proves irreducibility of $\mathcal{C}(2,n)$. The second establishes reducibility of $\mathcal{C}(k,n)$ for $k,n \geq 4$ through a simple argument using the previously mentioned matrix algebra connection, that is, we are quite easily able to find four matrices of any size greater than four that generate an algebra of dimension exceeding matrix size. The third is Guralnick's theorem on the reducibility of $\mathcal{C}(3,n)$ for $n \geq 29$, originally appearing in [6]. The proof of Guralnick's theorem has been built upon by multiple authors, and although at inception only cases $n \geq 32$ were covered, the proof can be adjusted as in [22] to use Weyr matrices, allowing us to cover the additional cases of $n = 29, 30, 31$. We explore the details of these adjustments in the fourth section of the third chapter. The final theorem covers only the case of $\mathcal{C}(3,4)$. The reason for including this case in particular, as well as it being the only other case which holds for $\mathcal{C}(3,4)$ over fields of arbitrary characteristic, is that after some examination, one realises that the methods for proving irreducibility of $\mathcal{C}(3,4)$ build upon the methods used to prove irreducibility of $\mathcal{C}(2,n)$ in a rather natural way, both requiring a proof that some set of *k-regular matrices* is dense in the variety.

The remaining known cases do receive a brief mention, as they are all individually proven through a different problem connected to matrix varieties - the *approximate simultaneous diagonalisation* problem. In fact, the paper [1] highlighting the connection between approximately simultaneously diagonalisable matrices and invariants of evolutionary models in biology was the initial motivator for several of the mathematicians currently researching these matrix variety problems.

In short, the ultimate aim of this report is to illustrate the link between problems concerning the dimension of a matrix algebra and problems concerning the reducibility of a matrix variety. En route, Weyr matrices will make frequent appearances, demonstrating the applicability of their use for research in these areas.

Chapter 1

The Weyr Form

A linear transformation on an n-dimensional vector space over a field F can be represented with respect to a given basis as an element of $M_n(F)$. A fundamental equivalence relation on square matrices is that of *similarity*, where matrices $A, B \in M_n(F)$ are similar if there exists $P \in \mathrm{GL}_n(F)$ with $A = P^{-1}BP$, that is, if they represent the same linear transformation with respect to different bases. Similarity preserves a multitude of properties of a matrix, such as determinant, eigenvalues and trace, so it stands to reason that one might want to find an exemplary representative for an equivalence class of matrices under similarity, and ideally a representative for which 'questions about any standard invariant relative to similarity can be immediately answered' [22, p36]. One can think of a *canonical form* as a convenient subset S of $M_n(F)$ such that every matrix is similar to a unique matrix in S. Jordan normal form is a well-known canonical form, and another, also with a multitude of applications, is called the *Weyr form*.

We dedicate the first section of this chapter to the definition of Weyr matrices. The distinguishing characteristics of Weyr matrices are described in the second section, where we focus primarily the block shifting effect that multiplication by a Weyr matrix has on a *blocked matrix*. This in turn gives rise to a practical description of the matrices that commute with a Weyr matrix. We proceed to establish the fact that Weyr matrices describe a canonical form in the third section. The chapter concludes by establishing duality of the Jordan and Weyr canonical forms. We mostly use results from Chapters 1, 2 and 3 of [22].

1.1 Weyr Matrices

Let $n, k \in \mathbb{N}$. Then a tuple $(n_1, \ldots, n_k)$ such that $n = n_1 + \cdots + n_k$ and $n_1 \geq \cdots \geq n_k$ is called a *partition* of n. Throughout the report we will be considering $n \times m$ matrices grouped into submatrices based on partitions of n and m. These will be referred to as *blocked* matrices (called *block* matrices in some texts). Grouping the entries in this way allows us to perform matrix multiplications in terms of the submatrices. In order for matrix multiplication to behave in this coherent way, the blocking of the matrices must be *conformable* [4, Definition 1.9.5]: if we want to multiply $A \in M_{a \times n}$ and $B \in M_{n \times b}$, then the partitions of n used for A and B must be equal. Since we will mainly be considering commuting square matrices, and therefore wish to be able to multiply them in either order, we always block square matrices according to the same partition both vertically and horizontally.

Example 1.1.1. We perform matrix multiplication in the standard way below.

$$\begin{pmatrix} 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 & 1 & 0 & 0 \\ 8 & -9 & 7 & 0 & 0 \\ 3 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & -1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -3 & 27 & 42 \\ 0 & 0 & 7 & 63 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 2 & 5 & -2 & 18 & 35 \\ 8 & -9 & 6 & 0 & 7 \end{pmatrix}$$

Alternatively, we can block the matrices according to the partition $(3, 2)$. Any blocks which are not displayed can be assumed to contain only zeros.

$$\begin{pmatrix} & A \\ B & C \end{pmatrix} := \left(\begin{array}{ccc|cc} & & & 3 & 6 \\ & & & 7 & 0 \\ & & & 1 & 0 \\ \hline 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right), \quad \begin{pmatrix} D & \\ E & F \end{pmatrix} := \left(\begin{array}{ccc|cc} 2 & 5 & 1 & & \\ 8 & -9 & 7 & & \\ 3 & 1 & 5 & & \\ \hline 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & -1 & 0 & 7 \end{array}\right)$$

Then to obtain the answer, we can multiply the matrices in terms of their blocks as below.

$$\begin{pmatrix} & A \\ B & C \end{pmatrix} \begin{pmatrix} D & \\ E & F \end{pmatrix} = \begin{pmatrix} AE & AF \\ BD + CE & CF \end{pmatrix}$$

We multiply the individual submatrices to obtain

$$AE = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{pmatrix}, \quad AF = \begin{pmatrix} 27 & 42 \\ 63 & 0 \\ 9 & 0 \end{pmatrix}, \quad BD = \begin{pmatrix} 2 & 5 & 1 \\ 8 & -9 & 7 \end{pmatrix},$$

$$CE = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -1 \end{pmatrix}, \quad CF = \begin{pmatrix} 18 & 35 \\ 0 & 7 \end{pmatrix},$$

and placing these into the matrix above gives the expected answer.

We also often consider *block-diagonal* matrices, where all blocks except those on the diagonal are zero. If A and B are square matrices, then we define the *direct sum* of A and B to be the block-diagonal matrix

$$A \oplus B := \begin{pmatrix} A & \\ & B \end{pmatrix}.$$

If we want to take the direct sum of several matrices $A_1, \ldots, A_s$, then we use the notation

$$\bigoplus_{i=1}^{s} A_i := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_s \end{pmatrix}.$$

Similarly *block upper-triangular* matrices will make an appearance. These are blocked matrices where all of the blocks below the diagonal are zero matrices.

We first introduce perhaps the most well-known of the canonical forms: *Jordan normal form.* If F is an algebraically closed field, then every square matrix with entries in F is similar to a unique Jordan matrix.

Definition 1.1.2 ([22, p38])**.** Let F be a field and $\lambda \in F$. A *Jordan block* of size n with eigenvalue λ is an $n \times n$ matrix with

- λ on the diagonal,
- 1 on the superdiagonal, and
- 0 elsewhere.

A *Jordan matrix* is a direct sum of Jordan blocks, where (by convention) blocks of the same eigenvalue are direct-summed in descending size order.

Given a Jordan matrix J, if $m_1 \geq n_2 \geq \cdots \geq m_l$ are the sizes of the Jordan blocks of J with eigenvalue λ, we define the *Jordan structure* of J associated to λ to be the tuple $(m_1, \ldots, m_l)$.

We call upon the reader to observe the parallels between the definition of the Jordan block and that of the *Weyr block.*

Definition 1.1.3 ([22, Definition 2.1.1])**.** Let F be a field and $\lambda \in F$. Let n be a positive integer and $(n_1, \ldots, n_s)$ a partition of n. The *Weyr block* with eigenvalue λ and *Weyr structure* $(n_1, \ldots, n_s)$ is the $n \times n$ matrix W blocked according to the partition $(n_1, \ldots, n_s)$ such that

- each diagonal block W_{ii} is equal to λI where I is the $n_i \times n_i$ identity matrix,

- each superdiagonal block $W_{(i,i+1)}$ is the $n_i \times n_{i+1}$ matrix of full column rank in row-reduced echelon form, known as a *generalised identity matrix*, and

- all other entries are zero.

Example 1.1.4. Below are the 8×8 Weyr blocks with eigenvalue 2 and Weyr structures $(4,2,2)$ and $(3,3,1,1)$ respectively. Notice that a generalised identity matrix is just an identity matrix with zero rows added to the bottom.

$$\left(\begin{array}{cccc|cc|cc} 2 & 0 & 0 & 0 & 1 & 0 & & \\ 0 & 2 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 2 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 2 & 0 & 0 & & \\ \hline & & & & 2 & 0 & 1 & 0 \\ & & & & 0 & 2 & 0 & 1 \\ \hline & & & & & & 2 & 0 \\ & & & & & & 0 & 2 \end{array}\right), \quad \left(\begin{array}{ccc|ccc|c|c} 2 & 0 & 0 & 1 & 0 & 0 & & \\ 0 & 2 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 2 & 0 & 0 & 1 & & \\ \hline & & & 2 & 0 & 0 & 1 & \\ & & & 0 & 2 & 0 & 0 & \\ & & & 0 & 0 & 2 & 0 & \\ \hline & & & & & & 2 & 1 \\ \hline & & & & & & & 2 \end{array}\right).$$

Definition 1.1.5. A *Weyr matrix* is a direct sum of Weyr blocks with **distinct**[1] eigenvalues.

A *nilpotent* matrix A is a matrix such that $A^n = 0$ for some $n \in \mathbb{N}$. Nilpotent matrices may only have eigenvalue 0, since if some nilpotent matrix A has a nonzero eigenvalue $\lambda \in F$ with corresponding (also nonzero) eigenvector v, then we have

$$0 = 0v = A^n v = \lambda^n v \neq 0,$$

which is a contradiction. Thus in the world of the Weyr form, a nilpotent Weyr matrix is a Weyr matrix with single eigenvalue 0, so is a Weyr block. For the problems we will consider, there exist some standard techniques which allow us to prove a statement about general matrices by considering only nilpotent matrices. We make frequent use of the following theorem, the details of which are are outlined in [22, Section 1.5].

Theorem 1.1.6 (Reduction to the Nilpotent Case, [22, Corollary 1.5.4])**.** *Let F be an algebraically closed field and $A \in M_n(F)$. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of A with corresponding algebraic multiplicities n_i. Then there exist nilpotent matrices $N_i \in M_{n_i}(F)$ such that A is similar to $(\lambda_1 I + N_1) \oplus \cdots \oplus (\lambda_k I + N_k)$.*

[1]This condition is necessary for the uniqueness result to hold. We will be in a position to further examine this after proving duality of the Jordan and Weyr forms.

1.2 The Centraliser of a Weyr Matrix

Later chapters will include applications of Weyr matrices, and most of these rely on one of two properties of the matrices. These are the *basis* and *block shifting effects*, giving rise to the form of the *centraliser* of a nilpotent Weyr matrix. This first remark follows [22, Theorem 2.4.1].

Remark 1.2.1. Let W be a nilpotent $n \times n$ Weyr matrix of Weyr structure $(n_1, \ldots, n_s)$. Consider the standard basis vectors $e_1, \ldots, e_n$. The first n_1 columns of W contain only zeros, so $We_i = 0$ for $1 \leq i \leq n_1$. Considering the following n_2 columns, that is, columns $n_1 + 1 \leq i \leq n_1 + n_2$, the ith column has a 1 in the $n_1 + i$th position and zeros elsewhere. Hence $We_{n_1+i} = e_i$ for $1 \leq i \leq n_2$. The general pattern is as follows: if we split the standard basis $\mathcal{B}$ into bases $\mathcal{B}_1, \ldots, \mathcal{B}_k$ with the first n_1 basis vectors in $\mathcal{B}_1$, the next n_2 basis vectors in $\mathcal{B}_2$, and so forth, then multiplication by W kills[2] the elements of $\mathcal{B}_1$ and sends the elements of $\mathcal{B}_{i+1}$ to $\mathcal{B}_i$, whilst preserving their order.

Example 1.2.2. The diagram below represents the action of multiplication by the nilpotent Weyr block of Weyr structure $(6, 5, 3, 2, 2)$ on the standard basis $e_1, \ldots, e_{18}$. Similar diagrams are used in [22, p39, p46-47].

$$
\begin{array}{cccccc}
 & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 & \mathcal{B}_4 & \mathcal{B}_5 \\
0 \leftarrow & e_1 \leftarrow & e_7 \leftarrow & e_{12} \leftarrow & e_{15} \leftarrow & e_{17} \\
0 \leftarrow & e_2 \leftarrow & e_8 \leftarrow & e_{13} \leftarrow & e_{16} \leftarrow & e_{18} \\
0 \leftarrow & e_3 \leftarrow & e_9 \leftarrow & e_{14} & & \\
0 \leftarrow & e_4 \leftarrow & e_{10} & & & \\
0 \leftarrow & e_5 \leftarrow & e_{11} & & & \\
0 \leftarrow & e_6 & & & &
\end{array}
$$

Note that the first column contains $n_1 = 6$ elements, the second contains $n_2 = 5$, the third contains $n_3 = 3$ and so on.

Similar is the block shifting effect that a nilpotent Weyr block W has on a matrix blocked according to the same partition as W. Because a Weyr block is like a Jordan block with 1's replaced with generalised identity matrices, we will first investigate the effect of multiplying by both Jordan blocks and generalised identity matrices.

[2] It should be noted that the phrase 'A kills b' is shorthand for 'A sends b to zero'.

Consider $\binom{I}{0}$, the generalised identity matrix of size $m \times n$ for $m \geq n$. We have

$$\begin{pmatrix} I \\ 0 \end{pmatrix} A = \begin{pmatrix} IA \\ 0A \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} B & C \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = BI + C0 = B,$$

for $A \in M_{n \times l}(F), B \in M_{l \times n}(F)$ and $C \in M_{l \times (m-n)}(F)$. That is, left multiplication adds $(m-n)$ zero rows to the bottom of a matrix, and right multiplication removes the rightmost $(m-n)$ columns of a matrix.

An $n \times n$ Jordan block J has entries $J_{ij} = \delta_{i+1,j}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the matrix product formula shows that if A is a matrix with which multiplication by J is compatible, then

$$(JA)_{ij} = \sum_{k=1}^{n} \delta_{i+1,k} A_{kj} = A_{i+1,j},$$

so left multiplication shifts elements upwards. Similarly

$$(AJ)_{ij} = \sum_{k=1}^{n} A_{ik} \delta_{k+1,j} = A_{i,j-1},$$

so right multiplication shifts elements to the right.

Combining these two effects allows us to observe the effect of left/right multiplication by a nilpotent Weyr block, like in [22, Example 2.3.2].

Example 1.2.3. Let W be the nilpotent Weyr block of Weyr structure $(3, 2, 1)$. Left multiplication by W shifts blocks upwards and adds zero rows to each block as one can observe below.

$$\left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ \hline & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ \cline{4-6} & & & & & 0 \end{array}\right) \left(\begin{array}{ccc|cc|c} 1 & 7 & 13 & 19 & 25 & 31 \\ 2 & 8 & 14 & 20 & 26 & 32 \\ 3 & 9 & 15 & 21 & 27 & 33 \\ \hline 4 & 10 & 16 & 22 & 28 & 34 \\ 5 & 11 & 17 & 23 & 29 & 35 \\ \hline 6 & 12 & 18 & 24 & 30 & 36 \end{array}\right) = \left(\begin{array}{ccc|cc|c} 4 & 10 & 16 & 22 & 28 & 34 \\ 5 & 11 & 17 & 23 & 29 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 6 & 12 & 18 & 24 & 30 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Right multiplication by W shifts blocks to the right and removes the right-most columns from each block.

$$\left(\begin{array}{ccc|cc|c} 1 & 7 & 13 & 19 & 25 & 31 \\ 2 & 8 & 14 & 20 & 26 & 32 \\ 3 & 9 & 15 & 21 & 27 & 33 \\ \hline 4 & 10 & 16 & 22 & 28 & 34 \\ 5 & 11 & 17 & 23 & 29 & 35 \\ \hline 6 & 12 & 18 & 24 & 30 & 36 \end{array}\right)\left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ \hline & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ \hline & & & & & 0 \end{array}\right) = \left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & 1 & 7 & 19 \\ 0 & 0 & 0 & 2 & 8 & 20 \\ 0 & 0 & 0 & 3 & 9 & 21 \\ \hline 0 & 0 & 0 & 4 & 10 & 22 \\ 0 & 0 & 0 & 5 & 11 & 23 \\ \hline 0 & 0 & 0 & 6 & 12 & 24 \end{array}\right)$$

Given these shifting effects, we can determine when a matrix commutes with a nilpotent Weyr block, or in other words, is contained in the *centraliser* of the Weyr block.

Definition 1.2.4 ([22, p96])**.** Let $A \in M_n(F)$. Then the *centraliser* of A, denoted $\mathcal{C}(A)$, is the set of matrices that commute with A.

A matrix commutes with a Weyr block exactly when shifting blocks upwards and adding zero rows coincides with shifting blocks rightwards and removing columns, as is encapsulated by the following theorem.

Theorem 1.2.5 ([22, Theorem 3.2.1])**.** *Let W be a nilpotent Weyr block, and let K be a matrix blocked according to the same partition as W, the (i, j)th block of which is denoted K_{ij}. Then $K \in \mathcal{C}(W)$ exactly when*

$$K_{ij} = \begin{pmatrix} K_{i+1,j+1} & * \\ 0 & * \end{pmatrix},$$

where $*$ denotes that the entries of the block may be freely chosen.

Note that $K \in \mathcal{C}(W)$ must be block upper-triangular. Comparison of the first column of blocks of the resulting matrices given by upward and rightward shifting gives us the equality $K_{i,1} = 0$ for $i > 1$ (compare with Example 1.2.3). The block $K_{i,j}$ for $i > j$ is contained in the top corner of the zero block $K_{i-(j-1),1}$, so is itself zero.

Remark 1.2.6. One may wonder how many free choices[3] we have for the entries of a matrix $K \in \mathcal{C}(W)$. This is answered in [22, Proposition 3.2.2] by ascending from the blocks in the lowest row to the blocks at the top. The bottom row has all blocks zero except the $n_s \times n_s$ block on the far-right, the entries of which can be freely chosen. The row above has all blocks zero except the $n_{s-1} \times n_{s-1}$ block and the $n_{s-1} \times n_s$ block. The first n_s columns of the $n_{s-1} \times n_{s-1}$ block are fixed, as the first n_s rows depend on K_{ss}, and the remaining entries are zero. Thus we have $n_{s-1}^2 - n_{s-1}n_s$ choices for this block, and $n_{s-1}n_s$ choices for the block on the right. Thus we have $n_{s-1}^2 + n_s^2$ choices so far. This pattern continues: we have $n_i^2 + \cdots + n_s^2$ free choices in rows $i, i+1, \ldots, s$, so we have $n_1^2 + \cdots + n_s^2$ free choices in total.

One consequence of the following lemma is that matrices commuting with a Weyr matrix $W_1 \oplus \cdots \oplus W_s$ are those of the form $K_1 \oplus \cdots \oplus K_s$ where each K_i commutes with W_i.

Lemma 1.2.7 ([22, Proposition 3.1.1])**.** *If A is of the form*

$$\bigoplus_{i=1}^{s} (\lambda_i I + N_i),$$

where $N_i \in M_{n_i}(F)$ is nilpotent, then matrices that commute with A are also of this form, that is, they have the same block-diagonal structure (but may have different eigenvalues and nilpotent parts).

1.3 Existence and Uniqueness of the Weyr Form

The two essential criteria on the canonical form checklist are that every square matrix is similar to some matrix in the canonical form, and that this matrix is unique (at least, up to some trivial equivalence). In this section, we show that Weyr matrices describe a canonical form for matrices in $M_n(F)$, with F algebraically closed. Thanks to Theorem 1.1.6, it suffices to show that every nilpotent matrix is similar to a Weyr block. Proving this requires the following tool.

Lemma 1.3.1 ([22, Lemma 2.2.1, (3)])**.** *If the first d columns of a matrix A are zero, then any elementary row operation that modifies only the first d rows of A can be realised as a conjugation by the corresponding elementary matrix.*

Proof. Let E be a matrix such that left multiplication by E is an elementary row operation. Then right multiplication by E^{-1} performs the inverse of the corresponding column operation. Since the first d columns of A are zero, right multiplication by E^{-1} has no effect. That is, $EAE^{-1} = EA$ as required. □

[3] This corresponds to the dimension of $\mathcal{C}(W)$ as an *algebra* over F: see Chapter 2

The proof of the following theorem proceeds by induction in a way that warrants more explanation than is granted at source in [22, Theorem 2.2.2]. In order to provide this, we describe an iterative process by which we can find a strictly[4] block upper-triangular matrix similar to a nilpotent matrix N.

Let N be a nilpotent matrix of size $n > 1$. We first claim that N has nonzero nullity. If N is the zero matrix then its nullity is equal to the matrix size, and if N is nonzero then there exists some r with $N^r \neq 0$ but $N^{r+1} = 0$. Thus there exists $v \in F^n$ with $N^r v \neq 0$. This means $N(N^r v) = N^{r+1} v = 0$, hence the nullity of N is at least 1.

Now we outline the process. Let N_1 be a nilpotent matrix of nullity n_1. Let $\mathcal{B}$ be an ordered basis of F^n such that the first n_1 vectors in $\mathcal{B}$ form a basis for the kernel of N_1. If P_1 is the matrix which changes from the standard basis to $\mathcal{B}$, then provided N_1 is not the zero matrix, the matrix $P_1^{-1} N_1 P_1$ is of the form

$$\begin{pmatrix} \overbrace{0}^{n_1} & X_{12} \\ 0 & N_2 \end{pmatrix}$$

since N_1 kills the first n_1 vectors in $\mathcal{B}$. The nth power of the above matrix is zero by nilpotency of N_1, and has N_2^n as its bottom right corner entry, so N_2 is itself a nilpotent matrix. By similar arguments as before, there exists an invertible matrix P_2 such that conjugation of $P_1^{-1} N_1 P_1$ by the matrix $I \oplus P_2$ gives a matrix of the form[5]

$$\begin{pmatrix} \overbrace{0}^{n_1} & \overbrace{X_{12}}^{n_2} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & N_3 \end{pmatrix}$$

where the matrix N_3 is nilpotent of strictly smaller size than N_2.

At each step, we find a nilpotent matrix N_{i+1} of strictly smaller size than N_i. Since our original matrix is of finite size, at some point the process terminates i.e. we reach a matrix N_s equal to the zero matrix, so all diagonal blocks of the resulting matrix are zero. The following proof proceeds by induction on the **number of steps** that this process requires for a given nilpotent matrix.

[4]By strictly block upper-triangular we mean that the diagonal blocks are zero matrices.

[5]In moving from the 2×2 blocked matrix to the 3×3 blocked matrix below, we abuse notation in that we have split X_{12} into the matrices X_{12} and X_{13}

Theorem 1.3.2 ([22, Theorem 2.2.2]). *Let F be a field and $N \in M_n(F)$ nilpotent. Then there exists a Weyr block similar to N.*

Proof. Assume that the matrix size n is greater than 1, since a nilpotent matrix of size 1 is just the zero matrix. If the process requires zero steps, then N must have been the zero matrix, which is the nilpotent Weyr block of Weyr structure (n), acting as our base case. Assume for the inductive step that if the process requires $s-2$ steps, then N is similar to a Weyr block with $(s-1)$ parts in its Weyr structure. Now suppose that the process requires $s-1$ steps for N. Let N have nullity n_1. Then N is similar to the matrix

$$\begin{pmatrix} \overbrace{0}^{n_1} & X_{12} \\ 0 & N' \end{pmatrix},$$

where N' is a nilpotent matrix for which the process requires $s-2$ steps. By the inductive hypothesis, N' is similar to a nilpotent Weyr block W. Let $(n_2, \dots, n_s)$ be its Weyr structure. Then N is similar to the matrix

$$\left(\begin{array}{c|c} 0 & X \\ 0 & W \end{array}\right) = \left(\begin{array}{c|ccccc} \overbrace{0}^{n_1} & \overbrace{X_{12}}^{n_2} & \overbrace{X_{13}}^{n_3} & \cdots & & \overbrace{X_{1s}}^{n_s} \\ \hline & 0 & I_3 & \cdots & & 0 \\ & & 0 & & & \vdots \\ & & & \ddots & & \\ & & & & 0 & I_s \\ & & & & & 0 \end{array}\right).$$

Using row operations to cancel the matrices $X_{13}, \dots, X_{1s}$ requires modifying only the first n_1 rows, and since the first n_1 columns are zero, Lemma 1.3.1 tells us this can be done through conjugation by elementary matrices. Hence N is similar to the matrix

$$\left(\begin{array}{c|ccccc} \overbrace{0}^{n_1} & \overbrace{X_{12}}^{n_2} & \overbrace{0}^{n_3} & \cdots & & \overbrace{0}^{n_s} \\ \hline & 0 & I_3 & \cdots & & 0 \\ & & 0 & & & \vdots \\ & & & \ddots & & \\ & & & & 0 & I_s \\ & & & & & 0 \end{array}\right).$$

Notice that X_{12} is an $n_1 \times n_2$ matrix. By the rank-nullity theorem, we have

$$\begin{aligned} n &= n_1 + \operatorname{rank} X_{12} + \operatorname{rank} I_3 + \cdots + \operatorname{rank} I_s \\ &= n_1 + \operatorname{rank} X_{12} + n_3 + \cdots + n_s, \end{aligned}$$

so $\operatorname{rank} X_{12} = n_2$, meaning X_{12} is a matrix of full column rank. Therefore there exist elementary row operations modifying only the first n_1 rows which place X_{12} in row-reduced echelon form, and again using Lemma 1.3.1, N is similar to the matrix

$$\begin{pmatrix} \overbrace{0}^{n_1} & \overbrace{I_2}^{n_2} & \overbrace{0}^{n_3} & \cdots & & \overbrace{0}^{n_s} \\ & 0 & I_3 & \cdots & & 0 \\ & & 0 & & & \vdots \\ & & & \ddots & & \\ & & & & 0 & I_s \\ & & & & & 0 \end{pmatrix}$$

which is the nilpotent Weyr block of Weyr structure $(n_1, n_2, \ldots, n_s)$. □

The following result appears at the beginning of [22, Theorem 2.2.2].

Corollary 1.3.3 (Existence of Weyr Form)**.** *Let F be an algebraically closed field. Then every $A \in M_n(F)$ is similar to a Weyr matrix.*

Proof. By Theorem 1.1.6, a matrix A with entries in an algebraically closed field is similar to some matrix $\bigoplus_{i=1}^{k}(\lambda_i I + N_i)$ with all λ_i distinct and N_i nilpotent. By the previous theorem, there exist invertible P_i such that each $P_i^{-1} N_i P_i$ is a nilpotent Weyr block W_i, so

$$\bigoplus_{i=1}^{k}(P_i^{-1}) \bigoplus_{i=1}^{k}(\lambda_i I + N_i) \bigoplus_{i=1}^{k}(P_i) = \bigoplus_{i=1}^{k}(\lambda_i I + W_i),$$

which conforms to the definition of a Weyr matrix. □

The proof of Theorem 1.3.2 establishes that the nullity of N is the first component of the Weyr structure of the corresponding Weyr block. This is integral to our uniqueness result.

Proposition 1.3.4 ([22, Proposition 2.2.3])**.** *If W is a Weyr block with eigenvalue λ and Weyr structure $(n_1, \ldots, n_s)$, then*

1. *s is the nilpotent index of $W - \lambda I$ i.e. the smallest positive integer s such that $(W - \lambda I)^s = 0$,*

2. *$n_1 = \text{nullity}(W - \lambda I)$, and*

3. *$n_i = \text{nullity}(W - \lambda I)^i - \text{nullity}(W - \lambda I)^{i-1}$.*

Proof. Statement (2) follows from the proof of Theorem 1.3.2. Statements (1) and (3) are consequences of the block shifting effect from Section 1.2, as the first $n_1 + \cdots + n_i$ columns of the matrix $(W - \lambda I)^i$ are zero and the rest are linearly independent. □

The Weyr structure of a Weyr block is determined by the nullities of its powers: values which are invariant with respect to similarity, giving us our uniqueness result.

Corollary 1.3.5 (Uniqueness of Weyr Form, [22, Theorem 2.2.4])**.** *All similar Weyr matrices are equal up to permutation of their Weyr blocks.*

The following definition is integrated into the result [22, Theorem 2.2.4].

Definition 1.3.6. Let $A \in M_n(F)$ for F an algebraically closed field. Then the *Weyr form* of A is the unique (up to permutation of Weyr blocks) Weyr matrix W similar to A . The *Weyr structure* of A is defined to be the Weyr structure of W.

1.4 Duality of Jordan and Weyr Matrices

The similarity of the Jordan and Weyr forms extends past the appearance of their definitions. Indeed there is a duality to the Jordan and Weyr structures of similar Jordan and Weyr matrices.

Definition 1.4.1 ([5, p327])**.** Let $\mathcal{N} = (n_1, \ldots, n_s)$ be a partition of n. Define the set $M_i = \{j \in \mathbb{N} : n_j \geq i\}$, and let $m_i = |M_i|$, equal to the number of parts of $\mathcal{N}$ of size i or greater. Let l be the largest integer such that $m_l \neq 0$. Then $\mathcal{M} = (m_1, \ldots, m_l)$ is the *dual partition* to $\mathcal{N}$.

From this definition, it is not immediately obvious that $\mathcal{M}$ should be a partition of n. Instead, the way this concept is often framed is in terms of *Young diagrams*. These are diagrams representing a partition $(n_1, \ldots, n_s)$ of n, consisting of s rows of squares, the ith row containing n_i squares. Consider Figure 1.1, which represents the partition $(5, 3, 2, 1)$ of 11. The dual partition is $(4, 3, 2, 1, 1)$. One can see that the ith part of the dual partition, m_i, is the number of squares in the ith column, whence the dual partition is also a partition of 11. Furthermore, this approach makes it clear that the dual of the dual of a partition is the original partition.

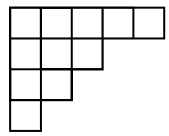

Figure 1.1: The Young diagram representing the partition $(5, 3, 2, 1)$ of 11.

Theorem 1.4.2 (Jordan-Weyr Duality, [22, Theorem 2.4.1])**.** *Let $A \in M_n(F)$ have Jordan structure $\mathcal{M} = (m_1, \ldots, m_l)$ associated to an eigenvalue λ. Then the Weyr structure of A associated to λ is the dual partition to $\mathcal{M}$.*

Proof. Let J be a nilpotent Jordan matrix of Jordan structure $(m_1, \ldots, m_l)$. Split the standard basis into sets $\mathcal{B}_1, \ldots, \mathcal{B}_l$ where $\mathcal{B}_1$ contains the first m_1 vectors, $\mathcal{B}_2$ contains the next m_2, and so forth. Multiplication by J kills the first vector of each $\mathcal{B}_i$, and sends the $(j+1)$th vector of $\mathcal{B}_i$ to the jth. Alternatively, given the dual partition $(n_1, \ldots, n_s)$ to $\mathcal{M}$, we can split the standard basis into sets $\mathcal{B}'_1, \ldots, \mathcal{B}'_s$ with $\left|\mathcal{B}'_j\right| = n_j$ such that $\mathcal{B}'_j$ contains the jth vector of each $\mathcal{B}_i$. Then multiplication by J kills the elements of $\mathcal{B}'_1$ and sends the elements of $\mathcal{B}'_{j+1}$ to $\mathcal{B}'_j$. This is exactly the action of the Weyr block W with Weyr structure $(n_1, \ldots, n_s)$ on the standard basis, so J and W represent the same linear transformation and are therefore similar matrices. □

The following example illustrates the same ideas as [22, Example .4.4]

Example 1.4.3. Consider the nilpotent Weyr block W of Weyr structure $(4, 2, 2)$. By Jordan-Weyr duality, the Jordan matrix J similar to W has Jordan structure $(3, 3, 1, 1)$. The matrix W acts on the standard basis as below, where arrows represent multiplication by W.

$$\begin{array}{ccccccccc}
 & & \mathcal{B}'_1 & & \mathcal{B}'_2 & & \mathcal{B}'_3 & & \\
0 & \leftarrow & e_1 & \leftarrow & e_5 & \leftarrow & e_7 & & \mathcal{B}_1 \\
0 & \leftarrow & e_2 & \leftarrow & e_6 & \leftarrow & e_8 & & \mathcal{B}_2 \\
0 & \leftarrow & e_3 & & & & & & \mathcal{B}_3 \\
0 & \leftarrow & e_4 & & & & & & \mathcal{B}_4
\end{array}$$

When the basis is ordered from top to bottom as above, the matrix representation of this linear transformation is W, but when reordered from left to right, the transformation is represented by J. Therefore the change of basis required to send W to J is

$$\begin{aligned} &e_1 \mapsto e_1, \quad e_2 \mapsto e_5, \quad e_3 \mapsto e_7, \quad e_4 \mapsto e_2, \\ &e_5 \mapsto e_6, \quad e_6 \mapsto e_8, \quad e_7 \mapsto e_3, \quad e_8 \mapsto e_4, \end{aligned}$$

so our change of basis matrix is given by P below, and conjugation by P gives us the rightmost matrix

$$P = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad P^{-1}WP = \left(\begin{array}{ccc|ccc|c|c} 0&1&0&&&&& \\ 0&0&1&&&&& \\ 0&0&0&&&&& \\ \hline &&&0&1&0&& \\ &&&0&0&1&& \\ &&&0&0&0&& \\ \hline &&&&&&0& \\ \hline &&&&&&&0 \end{array}\right),$$

which, as predicted, is the Jordan matrix of Jordan structure $(3, 3, 1, 1)$.

We are now in a position to consider the requirement of distinctness of eigenvalues in the definition of the Weyr matrix. Discarding this requirement would violate our uniqueness result, in that given Weyr blocks $W_1, \ldots, W_s$ with the same eigenvalue, there exists a single Weyr block W similar to $W_1 \oplus \cdots \oplus W_s$. However, there is more that we can say about the Weyr structure of W in terms of the structures of the W_i. This result as it relates to Weyr matrices was not found in the literature, but is a direct consequence of [5, Proposition 6] framed in terms of our particular canonical forms. Its usage in our scenario arose from a group session, during a discussion of the distinct eigenvalue condition.

Lemma 1.4.4. *Let W_1, W_2 be Weyr blocks of eigenvalue λ and Weyr structures $(n_1, \ldots, n_s)$ and $(m_1, \ldots, m_l)$ respectively, and assume $s \geq l$. Then $W_1 \oplus W_2$ has Weyr structure $(n_1 + m_1, \ldots, n_l + m_l, n_{l+1}, \ldots, n_s)$.*

Proof. There exist Jordan blocks J_1, J_2 and invertible matrices P_1, P_2 such that $W_i = P_i^{-1} J_i P_i$. Therefore

$$W_1 \oplus W_2 = (P_1 \oplus P_2)^{-1}(J_1 \oplus J_2)(P_1 \oplus P_2),$$

so the Jordan normal form of $W_1 \oplus W_2$ is $J_1 \oplus J_2$. Suppose J_1 and J_2 have respective Jordan structures $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_t)$, the dual partitions to the Weyr structures of W_1 and W_2. Then by definition, the Jordan structure of $J_1 \oplus J_2$ is the tuple

$$(a_1, \dots, a_r, b_1, \dots, b_t)$$

but with the parts sorted in order from greatest to least. Then by Jordan-Weyr duality, the Weyr structure of this matrix is the dual of this partition, which is exactly the tuple given in the statement of the lemma; a fact which can be seen using Young diagrams, but for the unsatisfied, is also the content of [5, Proposition 6]. □

Example 1.4.5. Consider the nilpotent Weyr blocks W_1, W_2 of Weyr structures $(2,1)$ and $(1,1,1)$ respectively.

$$W_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline & & 0 \end{array}\right), \quad W_2 = \left(\begin{array}{c|c|c} 0 & 1 & \\ \hline & 0 & 1 \\ \hline & & 0 \end{array}\right),$$

By Lemma 1.4.4, we expect $W_1 \oplus W_2$ to have Weyr structure $(2+1, 1+1, 1) = (3,2,1)$. By Jordan-Weyr duality, W_1 and W_2 are similar to Jordan matrices J_1, J_2 of Jordan structures $(2,1)$ and (3) respectively.

$$J_1 = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \quad J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

Hence $W_1 \oplus W_2$ is similar to $J_1 \oplus J_2$, which after permutation of Jordan blocks, can be seen to have Jordan structure $(3,2,1)$.

$$J_1 \oplus J_2 = \left(\begin{array}{cc|c|ccc} 0 & 1 & & & & \\ 0 & 0 & & & & \\ \hline & & 0 & & & \\ \hline & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & 1 & \\ & & & 0 & 0 & \\ \hline & & & & & 0 \end{array}\right).$$

This has dual partition $(3,2,1)$, so $W_1 \oplus W_2$ is similar to the Weyr block of Weyr structure

$$\left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ \hline & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ \hline & & & & & 0 \end{array}\right).$$

Chapter 2

Bounding the Dimension of a Matrix Algebra

A *matrix algebra* is an object that can be thought of as a subring of $M_n(F)$ with the additional structure of a vector space, meaning we can define its dimension as a vector space over F. If an algebra is generated by commuting matrices, then we can derive some results about its dimension, also giving rise to a 60 year old open problem central to our report: Gerstenhaber's problem.

The first section establishes the definition of an algebra in abstract terms disconnected from matrices, and introduces essential notions such as *algebra homomorphisms* and the *direct product* of algebras, before centering on matrix algebras and the language and notation we use for them specifically. Gerstenhaber's theorem on the dimension of a 2-generated commutative matrix algebra is covered in the second section, along with a slew of examples demonstrating that such bounds do not hold for $n \times n$ matrices in the k-generated case, $n, k \geq 4$. A recent computational attempt to disprove the three matrix analogue of Gerstenhaber's theorem uses Weyr matrices, and is outlined in the third section.

2.1 Matrix Algebras

Our development of algebras follows [14, Chapter 7].

Definition 2.1.1 ([14, Definition 1])**.** An (associative) algebra $\mathcal{A}$ over a field F is a pair consisting of a ring $(\mathcal{A}, +, \cdot, 0, 1)$ and a vector space $\mathcal{A}$ over F such that the underlying set $\mathcal{A}$, the addition, and the zero element are the same in the ring and the vector space, and

$$a(xy) = (ax)y = x(ay)$$

holds for all $a \in F, x, y \in \mathcal{A}$.

In this report, we assume the multiplication in all algebras to be commutative unless otherwise stated[1]. In [14, p409], a *subalgebra* of an algebra $\mathcal{A}$ is defined to be a subset of $\mathcal{A}$ which is also a subring and a vector subspace of $\mathcal{A}$. An *algebra homomorphism* is defined to be a function $f : \mathcal{A} \to \mathcal{B}$ between algebras $\mathcal{A}$ and $\mathcal{B}$ which is simultaneously a ring homomorphism and a linear map, a bijective algebra homomorphism is called an *algebra isomorphism*, and if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic algebras, we denote this by $\mathcal{A} \cong \mathcal{B}$.

Definition 2.1.2. Let $\mathcal{A}, \mathcal{B}$ be algebras. Then the *direct product* of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \times \mathcal{B}$, is the algebra consisting of pairs (a, b) with $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where addition, multiplication and scalar multiplication each occur component-wise. If we have algebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$, then we may write

$$\prod_{i=1}^{k} \mathcal{A}_i := \mathcal{A}_1 \times \cdots \times \mathcal{A}_k.$$

We say an algebra $\mathcal{A}$ *splits as a direct product* of algebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ when $\mathcal{A}$ is isomorphic to $\prod_{i=1}^{k} \mathcal{A}_i$. Then a basis for $\mathcal{A}$ is given by a union of bases of the $\mathcal{A}_i$, so $\dim \mathcal{A} = \sum_{i=1}^{k} \dim \mathcal{A}_i$.

We focus our attention on *matrix algebras*, which are subalgebras of $M_n(F)$. Let $A_1, \ldots, A_k$ be commuting $n \times n$ matrices. Then $F[A_1, \ldots, A_k]$ is defined to be the smallest subalgebra of $M_n(F)$ containing the matrices $A_1, \ldots, A_k$. Note that the identity matrix is contained in $F[A_1, \ldots, A_k]$ by our definition of an algebra. A matrix algebra which is equal to $F[A_1, \ldots, A_k]$ for some k is said to be *k-generated*[2].

Below is an example of finding the dimension of a commutative matrix algebra, which is my own work.

Example 2.1.3. Define

$$A = \begin{pmatrix} -9 & 4 & 0 & 2 \\ -20 & 9 & 0 & 4 \\ -10 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 5 & -1 & -5 & -3 \\ -5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[1]One notable exception is $M_n(F)$, which is of course a noncommutative algebra. However, when we refer to a subalgebra of $M_n(F)$, we assume the subalgebra is commutative unless otherwise stated.

[2]This definition has the peculiar consequence that an algebra containing only scalar matrices is 0-generated.

We want to find the dimension of the algebra $F[A, B]$. We have

$$AB = BA = C := \begin{pmatrix} 11 & -4 & -2 & -3 \\ 25 & -9 & -5 & -7 \\ 5 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now note that $A^2 = I$ and $B^2 = C$, so we have $AC = CA = BA^2 = B$ and $BC = CB = AB^2 = AC = B$. Thus $\{I, A, B, C\}$ spans $F[A, B]$. Finally, notice that $C = I - A + B$, so $\{I, A, B\}$ has the same span. Any linear combination of I and A has a zero in the $(1, 3)$ position, which is not the case for B. Thus $\{I, A, B\}$ is a basis, so $\dim F[A, B] = 3$.

Let $\mathcal{A}$ be an algebra containing only matrices of the form $A_1 \oplus \cdots \oplus A_s$, where $A_i \in M_{n_i}(F)$. If $\mathcal{A}$ contains the matrix with the $n_i \times n_i$ identity matrix in the ith diagonal block and zero blocks elsewhere, then we say we can *isolate* the ith block, as $\mathcal{A}$ automatically contains all matrices with A_i in the ith diagonal block and zero blocks elsewhere. If we can isolate all blocks, then it follows that $\mathcal{A}$ splits as a direct sum of the algebras $\mathcal{A}_i$ generated by the ith blocks of the matrices in $\mathcal{A}$, the isomorphism being

$$\mathcal{A}_1 \times \cdots \times \mathcal{A}_s \to \mathcal{A},$$
$$(A_1, \ldots, A_s) \mapsto A_1 \oplus \cdots \oplus A_s.$$

If $P \in \mathrm{GL}_n(F)$ and we set $B_i = P^{-1}A_iP$, then the algebras

$$F[A_1, \ldots, A_k], \;\; F[B_1, \ldots, B_k]$$

are isomorphic through the isomorphism $X \mapsto P^{-1}XP$. It follows that a set is a basis of an algebra if and only if its image under conjugation is a basis of the similar algebra, implying that dimension is a similarity invariant.

2.2 Gerstenhaber's Theorem and Generalisations

The reader should recognise the following theorem, proven for example in [16, Theorem 9.7]

Theorem 2.2.1 (Cayley-Hamilton)**.** *Let A be a matrix with characteristic polynomial p. Then $p(A) = 0$.*

The fact that $0 = p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I$ for some $a_i \in F$ gives us an expression of A^n as a linear combination of its lower powers. Therefore $\dim F[A] \leq n$, as the set $\{I, A, A^2, \ldots, A^{n-1}\}$ always spans $F[A]$. Note that this upper bound on the dimension is sharp, since $n \times n$ matrices with n linearly independent powers exist (consider a matrix which permutes the basis elements according to a permutation of order n).

An interesting 1-generated case is that of *1-regular* matrices. If $A \in M_n(F)$, then the *eigenspace* of A associated to an eigenvalue λ is the kernel of $(A - \lambda I)$, that is, the subspace of F^n consisting of all eigenvectors of A with eigenvalue λ.

Definition 2.2.2 ([22, p8])**.** Let F be an algebraically closed field. A matrix $A \in M_n(F)$ is *k-regular* if its eigenspaces have dimension $\leq k$. That is, if $\text{nullity}(A - \lambda I) \leq k$ for all eigenvalues λ of A.

Note that we only consider algebraically closed fields as matrices over other fields may have eigenvalues not contained in the field.

We will prove as in [22, Proposition 3.2.4] that the only matrices which commute with a 1-regular matrix A are polynomials in A. At source, the theorem is proven using the formula for the dimension of the centraliser of a Jordan matrix. However, after establishing the following lemma, the proof of which is adapted from [22, Proposition 5.1.1], we can prove the theorem directly using the form of the centraliser of a Weyr matrix. To prove the following lemma, we also require the observation that if $A \in M_n(F)$ has all eigenvalues in F, all of which are equal to λ, then $p(A)$ has single eigenvalue $p(\lambda)$. To see this, pass to an algebraically closed field and conjugate A into Weyr form W. Evaluate $p(W)$ and observe that since W is upper-triangular, each diagonal entry of $p(W)$ is $p(\lambda)$. Thus W has single eigenvalue $p(\lambda)$, and since $p(W) = p(P^{-1}AP) = P^{-1}p(A)P$, so does A.

Lemma 2.2.3. *Let F be a field and let s be a positive integer. Given distinct $\lambda_1, \ldots, \lambda_s \in F$, positive integers $n_1, \ldots, n_s$, and nilpotent matrices $N_i \in M_{n_i}(F)$, the algebra generated by the matrix*

$$A = \bigoplus_{i=1}^{s} (\lambda_i I + N_i)$$

contains all matrices I_i with the $n_i \times n_i$ identity matrix in the ith diagonal block and zero blocks elsewhere. That is, we can isolate all blocks of A, and therefore $F[A] \cong \prod_{j=1}^{s} F[N_j]$.

Proof. Fix i. Denote by $p_i(X)$ the characteristic polynomial of $\lambda_i I + N_i$, which we know to be $(X - \lambda_i)^{n_i}$ since λ_i is the single eigenvalue of this

$n_i \times n_i$ matrix. Note that $p_i(\lambda_j I + N_j) \neq 0$ has single eigenvalue $p_i(\lambda_j)$, which is nonzero as $\lambda_i \neq \lambda_j$. Hence $p_i(A)$ is equal to a matrix of the form

$$\bigoplus_{j=1}^{s} (p_i(\lambda_j)I + N_j'),$$

with each N_j' nilpotent and $N_i' = 0$. Note that this matrix has a zero matrix in the ith diagonal block and is nonzero elsewhere. Hence we can define the matrix

$$B_i = \prod_{i \neq j} p_j(A),$$

which is block-diagonal with ith diagonal block of the form

$$\prod_{i \neq j} p_j(\lambda_i I + N_i) = \left(\prod_{i \neq j} p_j(\lambda_i) \right) I + N_i''$$

for some nilpotent matrix N_i'', and the remaining diagonal blocks nonzero. Let λ denote the single eigenvalue of the ith block of B_i, which as established is nonzero. Then the B_i has characteristic polynomial $(X - \lambda)^{n_i}$. Define $q_i(X) = 1 - \frac{1}{(-\lambda)^{n_i}}(X - \lambda)^{n_i}$, which has zero constant coefficient. Therefore $q_i(B_i) = I_i$, that is, we can isolate the ith block for arbitrary i, so $F[A]$ splits as a direct product of the algebras $F[N_i]$. □

Theorem 2.2.4 ([22, Proposition 3.2.4])**.** *If A is 1-regular, then a matrix B commutes with A if and only if $B \in F[A]$. Moreover,* $\dim F[A] = n$.

Proof. It suffices to prove the theorem for the Weyr form of a 1-regular matrix, and we can further reduce to the case of a 1-regular nilpotent Weyr block by Lemma 1.2.7. Indeed, by the form of the centraliser of a Weyr block $W \in M_n(F)$, the matrices commuting with a 1-regular W are exactly the upper-triangular matrices with each diagonal constant. Note that there are n such diagonals. The matrix W^i has 1 along the ith diagonal and zeros elsewhere, the 0th diagonal being the main diagonal, the 1st being the superdiagonal, and so forth. Therefore $F[W]$ spans the space of matrices with each diagonal constant, which is n-dimensional. □

Remark 2.2.5. Matrices which are k-regular will make frequent appearances in the report, as there are simple criteria which allow us to determine the regularity of Weyr and Jordan matrices. If W is a Weyr matrix, then the nullity of $W - \lambda I$ is simply the nullity of $W_\lambda - \lambda I$, where W_λ is the

Weyr block of W with eigenvalue λ. The nullity of $W_\lambda - \lambda I$ is n_1: the first part of the Weyr structure of W associated to λ. Thus a Weyr matrix W is k-regular exactly when the Weyr structure associated to each eigenvalue has first part $n_1 \leq k$. By Jordan-Weyr duality, this means that for a Jordan matrix to be k-regular, the Jordan structure associated with λ should be of the form $(m_1, \ldots, m_l)$ where $l \leq k$, so a Jordan matrix is k-regular if there are at most k Jordan blocks with any given eigenvalue.

It follows from the Cayley-Hamilton theorem that the matrices of the form $A_1^{i_1} A_2^{i_2} \cdots A_k^{i_k}$ span $F[A_1, \ldots, A_k]$, immediately giving the upper bound of n^k on its dimension. A priori, there is no reason why this bound could not be attained, but a theorem of Schur shows otherwise.

Theorem 2.2.6 (Schur, [17])**.** *The maximum number of linearly independent commuting matrices of size n over a field is* $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$.

If we allow an arbitrary number of generators, then this bound is attained by the matrices I and e_{ij} (the matrix with a 1 in the (i,j) position and 0 elsewhere) for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq j \leq n$. However, if we restrict the number of generators of the algebra, then we can sometimes obtain better bounds. Case in point, the following theorem.

Theorem 2.2.7 (Gerstenhaber, [5, Theorem 2])**.** *Let $A, B \in M_n(F)$ be commuting matrices. Then* $\dim F[A,B] \leq n$.

This theorem admits a proof via purely linear-algebraic methods utilising the Weyr form, but we opt instead to prove the theorem in the third chapter using algebraic geometry.

We give our own quick example demonstrating that if we remove the commutativity requirement from the statement of Gerstenhaber's theorem, then the result falls down. For instance, consider the noncommutative algebra generated by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then A, B and the identity matrix are linearly independent, so $F[A,B]$ has dimension exceeding 2.

Similarly if we allow for sufficiently many (four) sufficiently large (4×4 at the smallest) matrices, the result does not hold. The following example follows [22, Theorem 6.3.4].

Example 2.2.8. By Theorem 2.2.6, we can generate a 5-dimensional matrix algebra using $e_{13}, e_{14}, e_{23}, e_{24} \in M_4(F)$, which is the highest dimensional subalgebra of $M_4(F)$ that can be achieved. For any larger matrix size we still need only four matrices to generate an algebra with dimension exceeding the matrix size. Consider for instance $(4+m) \times (4+m)$ matrices. Let A be an invertible matrix with m linearly independent powers (for instance, the permutation matrix corresponding to some m-cycle). Let $\mathcal{A}$ be the algebra generated by the matrices

$$\begin{pmatrix} e_{13} & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} e_{14} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e_{23} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e_{24} & 0 \\ 0 & 0 \end{pmatrix}.$$

Raising the first matrix to the power $2m$ gives us[3]

$$\begin{pmatrix} e_{13}^{2m} & 0 \\ 0 & A^{2m} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Thus we can isolate the second block, and subtracting this from the identity allows us to isolate the first block too. Therefore $\mathcal{A}$ splits as a direct product of $F[e_{13}, e_{14}, e_{23}, e_{24}]$ and $F[A]$, the former algebra having dimension 5 and the latter having dimension m due to the number of linearly independent powers. Thus the dimension of the generated algebra is $5+m$, which exceeds the matrix size.

A natural question is whether 4-generated matrix algebras violate greater bounds in terms of the matrix size. The paper [3] provides a method of constructing an algebra generated by four commuting $4m \times 4m$ matrices where the dimension of the algebra they generate is $5m$, demonstrating the failure of the bound $\dim F[A_1, A_2, A_3, A_4] < \frac{5}{4}n$ for $A_i \in M_n(F)$. In the paper, it is claimed that if $\mathcal{A}$ is the algebra generated by an upper-triangular matrix $D + U$ where D is a diagonal matrix and U is the strictly upper-triangular part, then $D, U \in \mathcal{A}$. This claim admits counterexamples even in the 2×2 case as was discovered during a group meeting: consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

which is present in $M_2(F)$ with F any field. This is a 1-regular matrix as it has distinct eigenvalues 0 and 1, and therefore by Theorem 2.2.4, the algebra

[3]The reason for raising to the power of $2m$ rather than simply m is for uniformity in the case $m = 1$, for which raising to the power of m would not give the desired result.

$F[A]$ that it generates is equal to its centraliser. But notice that

$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \text{ whereas } \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

that is, the strictly upper-triangular part does not commute with the matrix, so cannot be contained in the algebra $F[A]$. Fortunately the claim still holds due to Lemma 2.2.3.

Theorem 2.2.9 ([3, Section 3])**.** *For any $m \in \mathbb{N}$, there exists a 4-generated matrix algebra $\mathcal{A}$ with dimension exceeding the matrix size by m.*

Proof. Let F be a field containing at least m elements. Let $e_{13}, e_{14}, e_{23}, e_{24} \in M_4(F)$ as before. Select distinct $\lambda_i \in F$ and consider the matrices

$$E_{13} = \bigoplus_{i=1}^{m} \lambda_i I + e_{13}, \quad E_{14} = \bigoplus_{i=1}^{m} e_{14}, \quad E_{23} = \bigoplus_{i=1}^{m} e_{23}, \quad E_{24} = \bigoplus_{i=1}^{m} e_{24},$$

each of size $4m$. By Lemma 2.2.3, the algebra $\mathcal{A} = F[E_{13}, E_{14}, E_{23}, E_{24}]$ contains the matrices I_i for $1 \leq i \leq m$, and since the remaining generating matrices are block-diagonal with respect to the same partition as E_{13}, $\mathcal{A}$ splits as a direct product of m copies of the algebra $F[e_{13}, e_{14}, e_{23}, e_{24}]$ of dimension 5. Hence $\mathcal{A}$ has dimension $5m$ which exceeds the matrix size of $4m$ by m. $\square$

One caveat of this method not noted in Bergman's paper, is that if the underlying field has fewer than m elements, then Lemma 2.2.3 cannot be used to isolate each block, so the construction fails.

The following example is my own work, demonstrating Bergman's construction, as well as the block-isolating process integral to Lemma 2.2.3.

Example 2.2.10. Suppose we want to find complex matrices which generate an algebra that exceeds the matrix size by 3. Then we can take the four 12×12 matrices given by

$$E_{13} = \begin{pmatrix} I + e_{13} & 0 & 0 \\ 0 & 2I + e_{13} & 0 \\ 0 & 0 & 3I + e_{13} \end{pmatrix}, \quad E_{14} = \begin{pmatrix} e_{14} & 0 & 0 \\ 0 & e_{14} & 0 \\ 0 & 0 & e_{14} \end{pmatrix},$$

$$E_{23} = \begin{pmatrix} e_{23} & 0 & 0 \\ 0 & e_{23} & 0 \\ 0 & 0 & e_{23} \end{pmatrix}, \quad E_{24} = \begin{pmatrix} e_{24} & 0 & 0 \\ 0 & e_{24} & 0 \\ 0 & 0 & e_{24} \end{pmatrix},$$

where I is the 4×4 identity matrix. Consider the algebra

$$\mathcal{A} = F[E_{13}, E_{14}, E_{23}, E_{24}].$$

We give an explicit demonstration of how the third block is isolated. The characteristic polynomial of $I + e_{13}$ is $p_1(X) = (X-1)^4$, and the characteristic polynomial of $2I + e_{13}$ is $p_2(X) = (X-2)^4$. We have

$$p_1(E_{13}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (I+e_{13})^4 & 0 \\ 0 & 0 & (2I+e_{13})^4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I+4e_{13} & 0 \\ 0 & 0 & 16I+32e_{13} \end{pmatrix},$$

$$p_2(E_{13}) = \begin{pmatrix} (-I+e_{13})^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (I+e_{13})^4 \end{pmatrix} = \begin{pmatrix} I-4e_{13} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I+4e_{13} \end{pmatrix},$$

so to isolate the third block, we take the product

$$p_1(E_{13})p_2(E_{13}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16I+96e_{13} \end{pmatrix},$$

the third block of which has characteristic polynomial

$$(X-16)^4 = X^4 - 64X^3 + 1536X^2 - 16384X + 65536.$$

Therefore if we take

$$q_3(X) = \frac{-1}{65536}(X^4 - 64X^3 + 1536X^2 - 16384X)$$

we find that $I_3 = q_3(p_1(E_{13})p_2(E_{13}))$. We can perform similar computations to isolate the blocks I_1 and I_2, showing that $\mathcal{A}$ splits as the direct product $\prod_{i=1}^3 F[e_{13}, e_{14}, e_{23}, e_{24}]$, so the algebra has dimension $5 + 5 + 5 = 15$, exceeding the matrix size of 12 by 3 as required.

Gerstenhaber-like results have been seen to fail for 4-tuples of commuting matrices, and by virtue of this, fail for $(4+k)$-tuples as well (consider the fact that for a 4-generated counterexample with $A_i \in M_n(F)$, $\dim F[A_1, A_2, A_3, A_4, 0, \ldots, 0] = \dim F[A_1, A_2, A_3, A_4] \geq n$). This leaves only the case of triples, which is an open problem.

Problem 2.2.11 (Gerstenhaber's Problem)**.** *For all commuting $A, B, C \in M_n(F)$, is $\dim F[A, B, C] \leq n$?*

The upcoming section focuses on a new strategy for resolving this problem: hoping for a counterexample.

2.3 A Computational Strategy

In 2013, Holbrook and O'Meara created a MATLAB program designed to exploit the properties of Weyr matrices to search for a commuting triple generating an algebra of dimension exceeding matrix size. The inner workings of the program are described in [10]. We will illustrate the desirable properties of Weyr matrices which are applied in this scenario.

The authors of [10] make reference to the fact that Gerstenhaber's Problem can be reduced to the case of three nilpotent matrices commuting with a Weyr matrix, and we fill in the details here.

The vector space $M_{n\times m}(F)$ over F is nm-dimensional, and if we want to consider elements of $M_{n\times m}(F)$ as vectors in F^{nm}, we can use *vectorisation* [9, p291], which consists of mapping from a matrix to the vector obtained by stacking its columns as below

$$\operatorname{vec}\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ \vdots \\ a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}.$$

Lemma 2.3.1. *Let F be a field and $\overline{F}$ its algebraic closure. If there exist commuting $A, B, C \in M_n(F)$ with $\dim F[A, B, C] > n$ then there exist commuting $W, K, L \in M_m(\overline{F})$ with W a nilpotent Weyr block such that the dimension of $\overline{F}[W, K, L]$ exceeds m.*

Proof. Let $A, B, C \in M_n(F)$. Note that $\dim F[A, B, C]$ is determined by the rank of the matrix with columns the vectorisations of $A^iB^jC^k$, which is unchanged after passing to $\overline{F}$, so $\dim \overline{F}[A, B, C] = \dim F[A, B, C] > n$. We can place A in Weyr form

$$W = W_1 \oplus \cdots \oplus W_k,$$

where $W_i \in M_{n_i}(F)$, and simultaneously conjugate B, C to some K, L which have the same block-diagonal structure as W by Lemma 1.2.7. By Lemma 2.2.3, the algebra $\overline{F}[W, K, L]$ splits as a direct product of the algebras $\overline{F}[W_i, K_i, L_i]$, where $W_i, K_i, L_i \in M_{n_i}(F)$ and therefore

$$\sum_{i=1}^{k} n_i = n < \dim \overline{F}[W, K, L] = \sum_{i=1}^{k} \dim \overline{F}[W_i, K_i, L_i].$$

By pigeonholing, there must be some $\dim \overline{F}[W_i, K_i, L_i] > n_i$. $\square$

What are the benefits of reducing to the case of a nilpotent Weyr block? Computing the dimension of an algebra is generally a computationally expensive task. The problem of finding the number of linearly independent matrices of the form $A^iB^jC^k$ is equivalent to finding the rank of the matrix with columns the vectorisations of the matrices $A^iB^jC^k$ for $0 \leq i, j, k \leq n-1$. This is a matrix of dimension $n^2 \times n^3$. The authors of [10] believe that searching matrices of size 100×100 may be required before counterexamples begin to reveal themselves, and they remark that searching for a counterexample here requires computation of the ranks of several 10000×1000000 matrices. Using Weyr matrices removes some of this trouble through a dimension formula that depends on the *leading edge subspaces* of the algebra. The primary observation here is that a matrix in $\mathcal{C}(W)$ is determined by its top row of blocks. This means that if W is a nilpotent Weyr block, then given matrices $X_i \in M_{n_1 \times n_i}(F)$ with zeros in the appropriate places, the symbol $[X_1 \cdots X_s]$ uniquely determines the element of $\mathcal{C}(W)$ with top row $\big(X_1 \cdots X_s\big)$.

Definition 2.3.2 ([22, Definition 3.4.1])**.** Let $\mathcal{A}$ be an algebra containing a nilpotent Weyr block of Weyr structure $(n_1, \ldots, n_s)$. Then for $0 \leq i \leq s-1$, we define the *ith leading edge subspace* associated to $\mathcal{A}$ to be the vector space

$$U_i = \{X \in M_{n_1 \times n_{i+1}}(F) : [\overbrace{0 \cdots 0}^{i \text{ zeros}} X * \cdots *] \in \mathcal{A}\},$$

where $*$ denotes that the entries of the matrix may be freely chosen.

Theorem 2.3.3 (Leading Edge Dimension Formula, [22, Theorem 3.4.3])**.** *If $\mathcal{A}$ is an algebra containing a Weyr block of Weyr structure $(n_1, \ldots, n_s)$, then*

$$\dim \mathcal{A} = \sum_{i=0}^{s-1} \dim U_i.$$

Proof. All elements of $\mathcal{A}$ are determined by their top row of blocks. Consider the linear map $\pi_i : M_n(F) \to M_n(F)$ given by projecting a matrix blocked according to the partition $(n_1, \ldots, n_s)$ to the top $i \times i$ corner of blocks. That is,

$$\pi_i : \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix} \mapsto \begin{pmatrix} A_{11} & \cdots & A_{1i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ii} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Define $\mathcal{A}_i = \operatorname{im} \pi_i$. Restrict $\pi_{(s-1)}$ to $\mathcal{A}$. Since $A \in \mathcal{A}$ is determined by its top row of blocks, we have $A \in \ker \pi_{(s-1)}$ if and only if $A_{1i} = 0$ for $i \leq s-1$. Then $\ker \pi_{(s-1)} \mapsto A_{1s}$ is a linear bijection from $\ker \pi_{(s-1)}$ to $U_{(s-1)}$, so

$$\dim \mathcal{A} = \dim \mathcal{A}_{s-1} + \dim U_{s-1}.$$

If we restrict $\pi_{(s-2)}$ to $\mathcal{A}_{(s-1)}$, then we get a linear bijection from $\ker \pi_{(s-2)}$ to $U_{(s-2)}$, so

$$\dim \mathcal{A} = \dim \mathcal{A}_{s-2} + \dim U_{s-2} + \dim U_{s-1}.$$

Repeating this process gives

$$\dim \mathcal{A} = \dim \mathcal{A}_1 + \dim U_1 + \cdots + \dim U_{s-1},$$

and using the fact that $\mathcal{A}_1$ is isomorphic (as a vector space) to U_0 gives the required result. □

The remainder of this chapter gives an overview of `CommTriplesI` (the routine used to attempt to generate counterexamples), as well as an example of a single 'trial'. The example is genuine output from the program, which I have obtained by stepping through the MATLAB program (the source code of which was kindly provided by Kevin O'Meara) and recording the matrices that the program generates. The overview of the code is largely built from explanations given in [10], and [9].

A simplified overview of the routine `CommTriplesI` is as follows:

```
For s1 ≤ seed ≤ s2:
      Use seed to generate commuting top corner
      blocks K11, L11.
      For 2 ≤ k ≤ s:
           Attempt to generate K1k, L1k
           If generating is impossible then terminate
           the loop.
           Otherwise, calculate the dimensions of the
           leading edge subspaces U0, ··· Uk−1.
```

In the above setting, the user has provided a seed range $s_1, s_2 \in \mathbb{N}$ with $s_1 < s_2$, a Weyr structure $(n_1, \ldots, n_s)$ to base generation around, and a prime p which is used as the characteristic of the field over which arithmetic takes place, as well as other parameters determining how the program generates K_{11} and L_{11}.

Computing the dimension of the leading edge subspace U_i only requires finding the rank of a matrix whose columns are the vectorisations of the $(1, i+1)$ block of each matrix $W^i K^j L^k$, which is an $n_1 n_{i+1} \times n^3$ matrix, as opposed to the $n^2 \times n^3$ matrix mentioned earlier. The extension of K_{11}, L_{11} appears more complicated, but boils down to finding solutions to a linear system. The matrix determining the system arises from using the *Kronecker product* to convert the condition that the extended matrices commute into a linear system.

Definition 2.3.4 ([12, Definition 4.2.1])**.** Let $A \in M_{m\times n}(F), B \in M_{p\times q}(F)$. Then the matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

is called the **Kronecker product** of A and B.

The following Lemma can be proven by unravelling the relevant definitions.

Lemma 2.3.5 ([12, Lemma 4.3.1])**.** *Let $A \in M_{m\times n}(F), B \in M_{p\times q}(F), C \in M_{m\times q}(F)$ and $X \in M_{n\times p}(F)$. Then $AXB = C$ if and only if $(B^T \otimes A) \operatorname{vec} X = \operatorname{vec} C$.*

For a fixed Weyr block W of Weyr structure $(n_1, \ldots, n_s)$, denote by W_i the Weyr block of Weyr structure $(n_1, \ldots, n_i)$. To attempt to extend $K, L \in \mathcal{C}(W_{i-1})$ to $\tilde{K}, \tilde{L} \in \mathcal{C}(W_i)$, we require that the new blocks K_{1i}, L_{1i} satisfy the commutation requirement

$$K_{1*}L_{*i} = K_{11}L_{1i} + \cdots + K_{1i}L_{ii} = L_{11}K_{1i} + \cdots + L_{1i}K_{ii} = L_{1*}K_{*i},$$

noting that the matrices K_{ji}, L_{ji} are already known for $j > 1$ as they are determined by the top row. If we want to guarantee that $\tilde{K}, \tilde{L}$ can be extended to matrices in $\mathcal{C}(W)$, then we require that the new blocks K_{1i}, L_{1i} have zeros in specific locations. It follows that K_{1i}, L_{1i} arise as solutions to the linear system

$$\begin{pmatrix} M_1 & M_2 \\ D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \operatorname{vec} L_{1i} \\ \operatorname{vec} K_{1i} \end{pmatrix} = \begin{pmatrix} \operatorname{vec} Z \\ 0 \\ 0 \end{pmatrix},$$

where

- $M_1 = I_{n_i} \otimes K_{11} - K_{ii}^T \otimes I_{n_1}$ and $M_2 = -(I_{n_i} \otimes L_{11} - L_{ii}^T \otimes I_{n_1})$.
- $D \in M_{n_1 n_i}(F)$ is the diagonal matrix such that $D \operatorname{vec} K_{1i}$ returns the vector consisting of entries of K_{1i} which must be zero so that $\tilde{K}$ can possibly be extended to a matrix in $\mathcal{C}(W)$.
- $Z = (L_{1*}K_{*i} - L_{11}K_{1i} - L_{1i}K_{ii}) - (K_{1*}L_{*i} - K_{11}L_{1i} - K_{1i}L_{ii})$.

Example 2.3.6. We will use the above algorithm to generate a commuting triple (W, K, L) with W the nilpotent Weyr block of Weyr structure $(5, 4, 3)$, and check if it generates an algebra of dimension exceeding 12. We start by providing (among other things) a seed and some sparsity settings[4] to the routine `CommTriplesI`, as well as the prime $p = 5$ indicating the characteristic of the field over which to work. In response, the program generates the 5×5 blocks

$$K_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{11} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices both commute with W_1 (which is a zero block) and commute with each other. Note also that the bottom row of the first four columns of each matrix contains only zeros, as does the penultimate row of the first three columns, so these matrices have the potential to be extended to matrices commuting with W.

Next, we extend K and L to commuting matrices in $\mathcal{C}(W_2)$. The matrices determining the linear system are

- $M_1 = I_4 \otimes K_{11} - K_{22}^T \otimes I_5$ and $M_2 = -(I_4 \otimes L_{11} - L_{22}^T \otimes I_5)$.
- D has a 1 in the 5, 10, and 15th diagonal positions and zeros elsewhere.
- $Z = 0$.

The system can be solved, and the program selects the solutions

$$K_{12} = 0, \quad L_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[4] The particular settings used to generate these matrices are: $(p_0, p_1, p_2, p_c) = (0.5, 0.86, 0.94, 50)$ and `seed` $= 78$, with 5 being the only allowable 0th leading edge dimension.

The linear system governing extension to the third block is given by

- $M_1' = I_3 \otimes K_{11} - K_{33}^T \otimes I_5$ and $M_2' = -(I_3 \otimes L_{11} - L_{33}^T \otimes I_5)$
- $D' = 0$, as no further extension is required.
- $Z' = L_{12}K_{23} - K_{12}L_{23}$.

This system also has solutions, one of which is

$$K_{13} = 0, \quad L_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the matrices $K, L \in \mathcal{C}(W)$ with top rows

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \end{pmatrix}, \quad \begin{pmatrix} L_{11} & L_{12} & L_{13} \end{pmatrix}$$

form part of the commuting triple (W, K, L). To find the dimension of $(\mathbb{Z}/5)[W, K, L]$, we must compute the dimensions of leading edge subspaces U_0, U_1 and U_2. This involves finding the rank of matrices of size 25×12^3, 20×12^3 and 15×12^3 – a laborious task[5] which we leave to the computer. We obtain

$$\dim U_0 = 5, \quad \dim U_1 = 4, \quad \dim U_2 = 3,$$

the sum of which unfortunately does not violate the matrix size bound of 12. Better luck next time!

[5]In practice, these matrices will have many zero columns due to the selected sparsity settings. Furthermore, the program has subroutines which compute the dimension iteratively to avoid computing the rank of such matrices directly.

Chapter 3

Reducibility of Matrix Varieties

Theories surrounding algebraic and geometric objects combine in the field of algebraic geometry. The idea is that one can take a collection of polynomials $f_i \in F[x_1, \ldots, x_n]$, and study the geometric properties of the set consisting of points at which every f_i vanishes. Think of the case $F = \mathbb{R}$ and $n = 2$, where the set of roots of any polynomial in X and Y is a subset of the plane.

This set of n-tuples admits a striking topology, at which point we can use the language and theorems of topology to prove results about such vanishing sets, and in turn find information about algebraic properties determined by the vanishing of the polynomials. Since the commuting of $A, B \in M_n(F)$ is determined by the vanishing of the matrix $(AB - BA)$ which is a polynomial in the entries of A and B, we can use these tools to study tuples of commuting matrices, granting us access to an alternative route to study problems on the dimension of a matrix algebra.

We outline the objects of study, known as *affine varieties*, in the first section. The relationship between matrix varieties and matrix algebras centres around a topological property called *irreducibility*, which we develop in the second section. The long-awaited proof of Gerstenhaber's theorem is one application of irreducibility of a particular matrix variety, and we cover this in the third section. By showing that certain matrix varieties are irreducible, we can show that certain sizes of matrices satisfy the three matrix analogue of Gerstenhaber's theorem, so the fourth section covers these known cases. We conclude the chapter with a proof of Guralnick's theorem, which shows that all sufficiently large varieties of commuting triples of matrices are reducible, and hence we cannot use this avenue to fully resolve Gerstenhaber's problem. This proof exploits an invariant called the *dimension* of a variety. Out treatment of this concept amounts to the statement of some theorems prior to giving the proof of Guralnick's theorem. We include details about how the proof has changed over time to include more matrix sizes. For a majority of this section, we use Chapter 7 of [22] as a reference. Note that for Sections 3.3 to 3.5, we assume that all fields are algebraically closed.

3.1 Affine Varieties

Definition 3.1.1 ([22, p311])**.** Let F be a field. We define *affine n-space over F* to be the set of n-tuples of elements of F, and denote this by $\mathbb{A}^n$. We equip this with the *Zariski topology*, having closed subsets

$$V(S) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in S\},$$

where $S \subseteq F[x_1, \ldots, x_n]$. Such closed subsets are known as *affine varieties*.

Alternatively, the open subsets are of the form

$$U(S) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) \neq 0 \text{ for some } f \in S\}.$$

Paraphrasing [13, p2], the difference in notation between F^n and $\mathbb{A}^n$ comes from the implicit bundling of F^n with a vector space structure, which we forget when we consider $\mathbb{A}^n$.

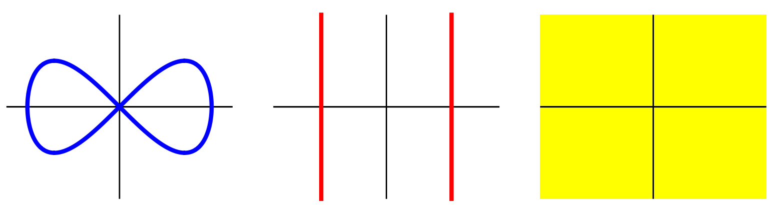

Figure 3.1: From left to right: varieties $V(x^4 + x^2 - y^2), V(x^2 - 1)$, and $V(0)$ depicted as subsets of the plane.[2]

Foundational theorems[3] of algebraic geometry allow us to assume that the subset S is finite.

If one has affine varieties $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ and a function $f : V \to W$, one can ask if f is represented by polynomials $f_1, \ldots, f_m \in F[x_1, \ldots, x_n]$, that is, $f(v) = (f_1(v), \ldots, f_m(v))$ for all $v \in V$. Such a function is called

[2] Diagrams made using Relplot [18].

[3] More specifically, we can consider the ideal I generated by all elements of S, which by Hilbert's basis theorem is finitely generated, so we can take S to be the set of generators.

a *polynomial map*, and a polynomial map with an inverse which is also a polynomial map is called an *isomorphism*.

Polynomial maps have nice properties with respect to the Zariski topology.

Lemma 3.1.2 ([22, Proposition 7.2.7])**.** *Polynomial maps are continuous with respect to the Zariski topology.*

Proof. It is sufficient to show that the preimage of a closed set under a polynomial map $g : \mathbb{A}^n \to \mathbb{A}^m$ is closed. A polynomial map is given by polynomials $g_1, \ldots, g_m \in F[x_1, \ldots, x_n]$. A closed subset V of $\mathbb{A}^m$ is determined by the vanishing of polynomials $f_1, \ldots, f_n \in F[x_1, \ldots, x_m]$, so the preimage of V is exactly the set of points at which the polynomials $f_i(g_1, \ldots, g_m)$ all vanish, i.e. the preimage is closed. □

Remark 3.1.3. In [22, p314], it is remarked that, in general, a polynomial map does not uniquely determine the polynomials $f_1, \ldots, f_m$, but little explanation is given, so we fill in the reasoning behind this here. Indeed, if a polynomial map $f : V \to W$ is represented by $f_1, \ldots, f_m \in F[x_1, \ldots x_n]$, and $g \in F[x_1, \ldots, x_n]$ is a polynomial which vanishes on all of V, then f is represented by polynomials of the form $f_i + gh_i$ for any choices of $h_i \in F[x_1, \ldots, x_n]$, since the term including h_i vanishes due to the presence of g.

Varieties need not appear to be tuples of points. Indeed, the set P_n of polynomials of degree at most n admits the structure of a variety through the following map:

$$(a_0, \ldots, a_n) \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

One can consider an $n \times n$ matrix with entries in F to be an n^2-tuple of elements in F through vectorisation. More generally, tuples $(A_1, \ldots, A_k)$ of such matrices can be identified with $\mathbb{A}^{kn^2}$, and any function which is a polynomial in the entries of $(A_1, \ldots, A_k)$ can be written as an element of $F[X_1, \ldots, X_{kn^2}]$. We give an example demonstrating how varieties and polynomial maps integrate with these matrix varieties, which is my own work. It somewhat generalises [22, Example 7.1.9].

Example 3.1.4. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Then the characteristic polynomial of A is given by

$$\det(\lambda I - A) = (\lambda - a_1)(\lambda - a_4) - a_2a_3 = \lambda^2 - (a_1 + a_4)\lambda + a_1a_4 - a_2a_3,$$

the coefficients of which are polynomials in the entries of A. Hence the map $f : M_2(F) \to \mathbb{A}^3$ sending a matrix to the coefficients of its characteristic polynomial is a polynomial map. More generally, the map $f : M_n(F) \times M_n(F) \to \mathbb{A}^{2n+2}$ sending a pair of matrices to the coefficients of their characteristic polynomials is a polynomial map. Consider the variety in $\mathbb{A}^{2n+1}$ determined by the vanishing of the polynomials

$$(x_1 - y_1), \ldots, (x_{n+1} - y_{n+1}) \in F[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

The preimage of this variety under f is the set of pairs of matrices with the same characteristic polynomial, so such a set is Zariski-closed.

Another relation determined by the vanishing of polynomials in the entries of a pair of matrices is commutativity, determined by the vanishing of the n^2 polynomials which are the entries of the matrix $AB - BA$. Thus the set of commuting k-tuples is an affine variety determined by the vanishing of the polynomials $A_iA_j - A_jA_i = 0$, where we run over all unordered pairs (i, j) with $1 \le i, j \le k$.

Definition 3.1.5 ([22, Example 7.1.5])**.** We call

$$\mathcal{C}(k, n) := \{(A_1, \ldots, A_k) : A_i \in M_n(F), A_iA_j = A_jA_i\}$$

the variety of *commuting k-tuples of $n \times n$ matrices over F*.

The following lemma encapsulates a result frequently used in [22, Chapter 7].

Lemma 3.1.6. *Let $f : W \to M_{r\times s}(F)$ be some polynomial function. Then the sets*

$$V = \{a \in W : \operatorname{rank} f(a) \le i\}, \quad U = \{a \in W : \operatorname{rank} f(a) \ge i\}$$

are Zariski-closed and Zariski-open respectively.

Proof. Let $A \in M_{r\times s}(F)$. Then $\operatorname{rank} A \le i$ if and only if the determinants of each of its $(i + 1) \times (i + 1)$ submatrices vanish. Similarly $\operatorname{rank} A \ge i$ if and only if M has an $i \times i$ submatrix with nonvanishing determinant. The determinant of a submatrix of A is a polynomial in the entries of A, so if S_i is the set of polynomials representing the determinants of all $i \times i$ submatrices of A, then V is the preimage of $V(S_{i+1})$ under f and U is the preimage of $U(S_i)$ under f, so V and U are closed and open respectively. □

Earlier we introduced k-regular matrices as matrices with eigenspaces of dimension bounded above by k, but an equivalent definition proves useful in showing that the set of k-regular matrices is a Zariski-open set. Literature on the properties of k-regular matrices is scarce, so the proof of the following result is my own work, and interestingly enough is an application of the Weyr form. The result itself is used in [20, Section 1], and a different proof likely exists in writing elsewhere.

Lemma 3.1.7. *Let F be an algebraically closed field. Then a matrix $A \in M_n(F)$ is k-regular if and only if there exist $v_1, \dots, v_k \in F^n$ such that the set $\{A^i v_j : 0 \le i \le n-1, 1 \le j \le k\}$ spans F^n.*[4]

Proof. For conciseness, if $A \in M_n(F)$ and

$$\mathcal{A} = \left\{A^i v_j : 0 \le i \le n-1, 1 \le j \le k\right\}$$

spans F^n, call $v_1, \dots, v_k$ an A-*basis* of F^n. Note that the span of $\mathcal{A}$ is equal to the span of $\{p(A)v_j : p \in F[X], 1 \le j \le k\}$.

We first prove the equivalence of the statements for Weyr blocks, and then extend the result to all matrices. Let W be a nilpotent Weyr block with Weyr structure $(n_1, \dots, n_s)$ which is k-regular but not $(k-1)$-regular. Then $n_1 = k$, and $n_i \le k$ for all i. Recall from Remark 1.2.1 that we can partition the standard basis into sets $\mathcal{B}_1, \dots, \mathcal{B}_s$ such that multiplication by W kills the elements of $\mathcal{B}_1$ and sends elements of $\mathcal{B}_{i+1}$ to $\mathcal{B}_i$. For each i, $|\mathcal{B}_i| = n_i \le k$. Let $v_{i,1}, \dots, v_{i,n_i}$ be the elements of each $\mathcal{B}_i$ labelled in such a way that $Wv_{i+1,j} = v_{i,j}$. Let $(m_1, \dots, m_k)$ be the dual partition to $(n_1, \dots, n_s)$. Define

$$w_j = v_{1,j} + \cdots + v_{m_j,j}.$$

We have

$$W^i w_j = v_{1,j} + \cdots + v_{m_j-i,j}.$$

Therefore $(W^i - W^{i+1})w_j = v_{m_j-i,j}$. Running over $1 \le j \le k$ and $0 \le i < m_j$ gives the desired result.

For the converse, assume that W is $(k+1)$-regular but not k-regular, and suppose that $w_1, \dots, w_k$ is a W-basis. The nullity of W is n_1, and since the vectors $W^i w_j$ for $1 \le i \le s-1$ are all contained in the image of W, the dimension of their span cannot exceed $n - n_1$. Thus $w_1, \dots, w_k$ alone must span an n_1-dimensional space, which is impossible as $k < k+1 = n_1$.

[4]This is equivalent to the statement that $v_1, \dots, v_k$ generate F^n as a $F[A]$-module.

To show the equivalence of the statements for a general matrix A, conjugate A into Weyr form $W = \bigoplus_{i=1}^{s}(\lambda_i I + W_i)$. Suppose A is k-regular. Then each Weyr block W_i must be k-regular, so there exists a W_i-basis $v_{i,1}, \ldots, v_{i,k} \in F^{n_i}$. View F^n as $F^{n_1} \times \cdots \times F^{n_s}$. By Lemma 2.2.3, we can express I_i, the matrix with the identity in the ith diagonal block and zero elsewhere, as $p_i(W)$. Thus

$$p_i(W)(v_{1,j} + \cdots + v_{s,j}) = v_{i,j},$$

and it follows that the vectors $v_{1,i} + \cdots + v_{s,i}$ for $1 \leq i \leq k$ give a W-basis for F^n. Changing the basis of the vectors gives us an A-basis for F^n

Conversely, if $v_1, \ldots, v_k$ is an A-basis, then after changing basis we obtain a W-basis. By Lemma 2.2.3 there exist polynomials in W which split the vectors into components $v_{i,1}, \ldots, v_{i,k}$ which generate F^{n_i} for each i, so each Weyr block is k-regular, meaning A is k-regular. □

Example 3.1.8. This example demonstrates how to find an A-basis of a regular matrix A, and is my own work. Let $A = P^{-1}WP$, where W is the Weyr matrix

$$\begin{pmatrix} W_1 & \\ & W_2 \end{pmatrix} = \left(\begin{array}{ccc|cc|c|c|c} 1 & 0 & 0 & 1 & 0 & & & \\ 0 & 1 & 0 & 0 & 1 & & & \\ 0 & 0 & 1 & 0 & 0 & & & \\ \hline & & & 1 & 0 & & & \\ & & & 0 & 1 & & & \\ \hline & & & & & -1 & 1 & \\ \hline & & & & & & -1 & 1 \\ \hline & & & & & & & -1 \end{array}\right),$$

which has Weyr structure $(3,2)$ associated to the eigenvalue 1 and $(1,1,1)$ associated to the eigenvalue -1. We can see that W is 3-regular, so we can find a W-basis. The vectors e_1+e_4, e_2+e_5, e_3 form a W_1-basis and $e_6+e_7+e_8$ on its own is a W_2-basis. Therefore $e_1 + e_4 + e_6 + e_7 + e_8, e_2 + e_5, e_3$ is a W-basis. Applying the change of basis represented by P gives us an A-basis.

Lemma 3.1.9 ([20, Proposition 1])**.** *Let F be an algebraically closed field. The set of k-regular matrices is open in $M_n(F)$.*

Proof. Fix some k-tuple $\mathcal{V} = (v_1, \ldots, v_k)$, with each $v_i \in F^n$. Given a matrix A, construct the $n \times kn$ matrix $M(A, \mathcal{V})$ where the columns are the vectors $A^i v_j$ running over all i, j. Then the function $A \mapsto M(A, \mathcal{V})$ is a polynomial map. The vectors $A^i v_j$ span F^n exactly when $M(A, \mathcal{V})$ has rank n, so by

Lemma 3.1.6, the set of matrices A such that the $A^i v_j$ span F^n is open. The set of k-regular matrices is the union over all $\mathcal{V}$ of these open sets, so is itself open. $\square$

In view of this, any subset of $\mathcal{C}(k, n)$ determined by the condition that the ith matrix of a tuple be j-regular is also open, since it is the preimage of the set of j-regular matrices under the projection to the ith element, which is a polynomial map.

3.2 Reducibility

The following property is essential for our treatment of matrix varieties in later sections.

Definition 3.2.1 ([22, Definition 7.4.1])**.** Let $V \subseteq \mathbb{A}^n$. We call V *reducible* if there exist proper closed subsets C_1, C_2 of V such that $V = C_1 \cup C_2$, and *irreducible* otherwise.

An equivalent condition for a set V being irreducible is that each pair U_1, U_2 of nonempty open subsets of V has nonempty intersection, or equally that every open subset of V is *dense* in V. Note that these definitions extend to arbitrary topological spaces.

The first part of this example explains a remark in [22, p323], and the second is my own example of a reducible variety.

Example 3.2.2.

1. It is apparent that any Hausdorff topological space X with more than one point is reducible, since for any $x, y \in X$ we can find disjoint open sets U, V containing x and y respectively, and taking complements gives the desired result. For instance, $\mathbb{R}$ with the standard topology can be written as $(-\infty, 1] \cup [-1, \infty)$.

2. The variety V over $\mathbb{C}$ determined by the vanishing of the polynomial $(x^4y^4 - 4)$ is reducible. Indeed, we can consider the the closed subsets $V \cap V(x^2y^2 - 2)$ and $V \cap V(x^2y^2 + 2)$ of V. Every point of V is contained in one of these closed sets, as $(x^4y^4 - 4) = (x^2y^2 - 2)(x^2y^2 + 2)$. Moreover the closed sets are disjoint and nonempty, so are proper subsets of V.

The book [22, Chapter 7] includes some examples of prominent subgroups of $M_n(F)$ and methods for determining their reducibility. In the spirit of this, we give an example of our own that lies in a similar vein, but with a family of matrix groups not mentioned in the text.

Example 3.2.3. Let F be an algebraically closed field, and $(\cdot,\cdot) : F^n \times F^n \to F$ a bilinear form on F^n, that is, a map which is linear in both variables. What algebro-geometric properties has the matrix group

$$V = \{A \in M_n(F) : (Av, Aw) = (v, w) \text{ for all } v, w \in F^n\}?$$

First, is V a variety? We can represent the bilinear form by a matrix $B \in M_n(F)$ so that

$$(v, w) = w^T Bv$$

for all $v, w \in F^n$. Then for a matrix A, preservation of the bilinear form is equivalent to

$$A^T BA = B,$$

that is, V is determined by the vanishing of $A^T BA - B$, so V is a variety.

We give some cases in which V can be seen to be reducible. One necessary condition for $A^T BA = B$ is that $\det(A)^2 \det(B) = \det(B)$, which, for $\det B \neq 0$, requires that $\det(A)^2 = 1$. Hence if n is odd and F has characteristic greater than 2, the matrices I and $-I$ both lie in V, and so the disjoint closed sets $V \cap V(\det A - 1)$ and $V \cap V(\det A + 1)$ which cover V are nonempty. Therefore in this case, V is reducible.

Lemma 3.2.4 ([22, Propositions 7.2.6, 7.4.2])**.** *Over an infinite field, $\mathbb{A}^n$ is irreducible.*

Proof. We first prove that a polynomial vanishes on all of $\mathbb{A}^n$ over an infinite field if and only if the polynomial is zero. The statement is trivially true for constant polynomials. A single variable polynomial of positive degree has finitely many roots by the fundamental theorem of algebra acting as our base case. We will perform an inductive step to prove the statement for all polynomials.

Let $f \in F[x_1, \ldots, x_{n+1}]$ and assume without loss of generality that the degree of x_{n+1} is $k \geq 1$. Then we can write

$$f(x_1, \ldots, x_{n+1}) = f_0 + x_{n+1} f_1 + \cdots + x_{n+1}^k f_k,$$

where each $f_i \in F[x_1, \ldots, x_n]$, and f_k is nonzero. Fixing $a_1, \ldots, a_n$ with $f_k(a_1, \ldots, a_n) \neq 0$, which is permissible by the induction hypothesis, gives us a nonzero polynomial $f(a_1, \ldots, a_n, x_{n+1})$ in a single variable, which has finitely many roots, so must have infinite nonvanishing set, and in particular, does not vanish on all of $\mathbb{A}^n$.

It suffices to show that $U(f) \cap U(g)$ is nonempty for nonempty $U(f), U(g)$, as sets of this form give a basis for the open subsets of $\mathbb{A}^n$. Assume $f, g \neq 0$. Since $F[x_1, \ldots, x_n]$ is an integral domain, $fg \neq 0$, so fg does not vanish everywhere by the above claim, and therefore $U(fg) \subseteq U(f) \cap U(g)$ is nonzero. □

Immediate from Lemma 3.2.4 is the fact that $M_n(F)$ and P_n are irreducible for matrix entries/coefficients in an infinite field. This is essential for the proofs of (ir)reducibility of $\mathcal{C}(k, n)$, so from now on, we assume that all fields are algebraically closed, and therefore infinite, unless otherwise stated.

The following results are purely topological, and we state them without proof.

Lemma 3.2.5 ([22, Propositions 7.4.3, 7.4.14])**.**

1. *The continuous image of an irreducible set is irreducible.*
2. *$V \subseteq \mathbb{A}^n$ is irreducible if and only if its closure in $\mathbb{A}^n$, denoted $\overline{V}$, is irreducible.*

Remark 3.2.6. One consequence of part 1 of the above lemma is that a line between two points $a, b \in \mathbb{A}^n$ is irreducible. This follows from the fact that

$$\begin{aligned} f : \mathbb{A} &\to \mathbb{A}^n, \\ t &\mapsto at + b(1-t), \end{aligned}$$

is a polynomial map with irreducible domain, so has irreducible image. We use this fact frequently in the following two sections.

If $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$, then we can define the product $V \times W$ to be the usual Cartesian product of V and W as sets, but with topology the subspace topology of $\mathbb{A}^{n+m}$. We stress as in [22, Remark 7.2.8] that this is not equal to the set $V \times W$ with the product topology, though projections $V \times W \to V$ and inclusions $v \mapsto (v, w)$ for fixed w are still polynomial maps: a property that the following result relies on, making it true for general topological spaces with the product topology as well.

Lemma 3.2.7 ([22, 7.4.16])**.** *Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be irreducible. Then $V \times W$ is irreducible.*

The following theorem is essential to the definition of the *dimension* of a variety, as we will see in Section 3.5. We state it without proof.

Theorem 3.2.8. *Let $V \subseteq \mathbb{A}^n$. Then there exist unique irreducible sets $V_1, \ldots, V_k \subseteq \mathbb{A}^n$ such that $V = V_1 \cup \cdots \cup V_k$. We call the V_i the irreducible components of V. Note that the components are not necessarily disjoint.*

Example 3.2.9. Let $U \subseteq M_n(F)$ be the set of 1-regular matrices. Consider the subset X_i of $\mathcal{C}(k, n)$ where the ith matrix is 1-regular. Denote by X the intersection of all such sets, so X is the subset of $\mathcal{C}(k, n)$ where all matrices in each tuple are 1-regular. We will show irreducibility of $\overline{X_1}$ using the same method as the proof of [22, Theorem 7.6.1] and use this to show irreducibility of all $\overline{X_i}$ through irreducibility of $\overline{X}$ in the same way as [20, Lemma 11].

The domain of the polynomial function

$$f : M_n(F) \times P_n^{k-1} \to \mathcal{C}(k, n),$$
$$f(A, p_1, \ldots, p_{k-1}) = (A, p_1(A), \ldots, p_{k-1}(A)),$$

is irreducible, and therefore so is its image. The fact that X_1 is open follows from Lemma 3.1.9, and in view of Theorem 2.2.4, X_1 is contained in $\operatorname{im} f$. Therefore X_1 is dense in $\operatorname{im} f$, that is $\overline{X_1} = \operatorname{im} f$, so $\overline{X_1}$ is irreducible.

Now note that $X \subseteq X_i$, so $\overline{X} \subseteq \overline{X_i}$. We want to show equality, so we will show the reverse inclusion. Fix $(A_1, \ldots, A_k) \in X_i$. Then the line L defined by the image of the polynomial map

$$f : \mathbb{A} \to \mathcal{C}(k, n)$$
$$t \mapsto (A_1 t + A_i(1 - t), \ldots, A_k t + A_i(1 - t)),$$

intersects X at $t = 0$. Since X is open as it is the intersection of k open sets, $X \cap L$ is dense in L, so we have

$$(A_1, \ldots, A_k) \in L = \overline{X \cap L} \subseteq \overline{X},$$

and since $(A_1, \ldots, A_k)$ was chosen arbitrarily, we have $X_i \subseteq \overline{X}$, so $\overline{X_i} = \overline{X}$. Irreducibility of $\overline{X}$, and therefore irreducibility of all $\overline{X_i}$, follows from this.

Remark 3.2.10. In fact, if we were to define X_i to be the subset of $\mathcal{C}(k, n)$ with the ith matrix k-regular, then the same argument used above establishes that $\overline{X_i} = \overline{X}$ for all i (though (ir)reducibility of $\overline{X}$ and $\overline{X_i}$ cannot be determined by this argument, as a k-regular matrix A may commute with a matrix outside of $F[A]$ if $k \geq 2$).

The sets of i-regular matrices will prove indispensable when studying irreducibility of the matrix varieties $\mathcal{C}(k, n)$. If we fix some $A \in M_n(F)$, then we can consider a related variety, $\mathcal{C}(k, A)$, defined as in [7] to be the set of commuting k-tuples of matrices $A_1, \ldots, A_k$ such that each A_i commutes with A. This is a variety, and we may wish to consider whether it is reducible.

Theorem 3.2.11. *If A is a 3-regular matrix, then $\mathcal{C}(k, A)$ is irreducible.*

Note that all 1- and 2-regular matrices are also 3-regular by our definition. Proofs for 2-regular and 3-regular matrices can be found in [20, Corollary 10] and [26, Theorem 12] respectively, and for 1-regular matrices the proof follows from the isomorphism of $\mathcal{C}(k, A)$ with the irreducible variety P_n^k. The varieties $\mathcal{C}(2, A)$ influence our knowledge of where to check for counterexamples to Gerstenhaber's problem via the following theorem.

Theorem 3.2.12 ([26, Corollary 20])**.** *If $\mathcal{C}(2, A)$ is irreducible and $(B, C) \in \mathcal{C}(2, A)$, then $\dim F[A, B, C] \leq n$.*

3.3 Gerstenhaber's Theorem via Algebraic Geometry

The following theorem cements the connection between Gerstenhaber-like problems and the problem of reducibility of matrix varieties.

Theorem 3.3.1 ([22, Proposition 7.6.5])**.** *Consider $\mathcal{C}(k, n)$ over F. If $\mathcal{C}(k, n)$ is irreducible then for all commuting $A_1, \ldots, A_k \in M_n(F)$, we have $\dim F[A_1, \ldots, A_k] \leq n$.*

Proof. Let $W = \{(A_1, \ldots, A_k) \in \mathcal{C}(k, n) : \dim F[A_1, \ldots, A_k] \leq n\}$. Since a 1-regular matrix A commutes only with polynomials in A, W contains the set X_1. Since $\mathcal{C}(k, n)$ is irreducible, X_1 is dense in $\mathcal{C}(k, n)$, so if we can show that W is closed then we have

$$\mathcal{C}(k, n) = \overline{X_1} \subseteq \overline{W} = W \subseteq \mathcal{C}(k, n),$$

so $W = \mathcal{C}(k, n)$, which proves the theorem.

To show that W is closed, consider the polynomial map sending the matrices $A_1, \ldots, A_k$ to the $(n^2 \times n^k)$ matrix whose columns are the vectorisations of the products $A_1^{i_1} \cdots A_k^{i_k}$, where $0 \leq i_j \leq n - 1$. The dimension of $F[A_1, \ldots, A_k]$ is the rank of $M(A_1, \ldots, A_k)$, so the fact that W is closed follows from Lemma 3.1.6. □

Example 2.2.8 together with Theorem 3.3.1 therefore serves as a proof that $\mathcal{C}(k,n)$ is reducible for $k,n \geq 4$. It is through the route unlocked by Theorem 3.3.1 that we prove Gerstenhaber's theorem, and give a partial answer to Gerstenhaber's problem for particular matrix sizes.

One of the earliest results on the reducibility of $\mathcal{C}(k,n)$ is due to Motzkin and Taussky.

Theorem 3.3.2 (Motzkin-Taussky, [22, Theorem 7.6.1])**.** *$\mathcal{C}(2,n)$ is irreducible for all n.*

The strategy is to show that the set X_1 is dense in $\mathcal{C}(2,n)$.

Lemma 3.3.3. *Let $A \in M_n(F)$. Then there exists a 1-regular matrix which commutes with A.*

Proof. Let $J_1 \oplus \cdots \oplus J_k$ be the Jordan normal form of A, where the block J_i has eigenvalue a_i. If we choose distinct $b_i \in F$, then the matrix

$$B = (J_1 + (b_i - a_i)I) \oplus \cdots (J_k + (b_k - a_k)I)$$

commutes with A, and is a Jordan matrix with exactly one block corresponding to each eigenvalue b_i, so by Remark 2.2.5, B is 1-regular. □

The following proof differs from that given in [22] in that we instead use the irreducibility of lines to prove that the set of matrices with the first matrix 1-regular is dense in $\mathcal{C}(2,n)$. This technique is used multiple times in [7].

Proof (of Motzkin-Taussky). Fix $(A,B) \in \mathcal{C}(2,n)$. Then B commutes with a 1-regular matrix, call it R. Therefore the line L determined by the image of the polynomial function $t \mapsto (tA + (1-t)R, B)$ intersects X_1 at $t = 0$. Lines are irreducible, so X_1 is dense in L, meaning $(A,B) \in L = \overline{X_1 \cap L} \subseteq \overline{X_1}$. Since (A,B) was arbitrary, $\mathcal{C}(2,n) = \overline{X_1}$. Example 3.2.9 establishes irreducibility of $\overline{X_1}$, which in turn establishes irreducibility of $\mathcal{C}(2,n)$. □

3.4 Cases of Irreducibility of $\mathcal{C}(3,n)$

The variety $\mathcal{C}(3,n)$ is known to be irreducible for $1 \leq n \leq 4$ over algebraically closed fields of arbitrary characteristic, and for $5 \leq n \leq 10$ over algebraically closed fields of characteristic zero[5]. The root of this is the difference of techniques: cases 1 through 4 are proven through the framework of algebraic geometry, and cases 5 through 10 are proven by way of a different linked problem.

Proving that $\mathcal{C}(3, n)$ is irreducible for small matrices requires us to consider *maximal* subalgebras.

Definition 3.4.1 ([22, p221])**.** Let $\mathcal{A}$ be a proper subalgebra of $\mathcal{B}$. We call $\mathcal{A}$ a *maximal* subalgebra of $\mathcal{B}$ if, for any other proper subalgebra $\mathcal{A}'$ of $\mathcal{B}$, $\mathcal{A} \subseteq \mathcal{A}'$ implies $\mathcal{A} = \mathcal{A}'$.

We state the following theorem without proof, and refer the reader to the papers [15] and [19] in which it was independently proven.

Theorem 3.4.2 (Laffey-Lazarus; Neubauer-Saltman)**.** *Let $A, B \in M_n(F)$. Then $F[A, B]$ is a maximal subalgebra of $M_n(F)$ if and only if $\dim F[A, B] = n$.*

Lemma 3.4.3 ([22, Example 5.4.5])**.** *Let $\mathcal{A}$ be a subalgebra of $M_n(F)$ for $n \leq 3$. Then $\mathcal{A} = F[X, Y]$ for some $X, Y \in \mathcal{A}$.*

Proof. Suppose the algebra $\mathcal{A}$ is not 2-generated, and therefore contains matrices I, X, Y, Z where I is the identity and $Z \notin F[X, Y]$ The subalgebra $F[X, Y]$ of $\mathcal{A}$ has dimension 3 as it contains the linearly independent matrices I, X, Y. For $n = 1, 2$ we have already reached a contradiction, as $\dim F[X, Y] \leq n$ by Theorem 2.2.7, whereas for $n = 3$, Theorem 2.2.7 tells us that $\dim F[X, Y] = 3$, and therefore by Theorem 3.4.2 $F[X, Y]$ must be a maximal subalgebra of $M_n(F)$, which is a contradiction as $F[X, Y] \subsetneq F[X, Y, Z]$). □

For small matrices, the proof is uniform and even holds for tuples containing any number of matrices.

Theorem 3.4.4 ([22, Corollary 7.6.7])**.** *$\mathcal{C}(k, n)$ is irreducible for $n \leq 3$.*

Proof. Let $A_1, \ldots, A_k$ be a commuting k-tuple of $n \times n$ matrices. It follows from Lemma 3.4.3 that the algebra $F[A_1, \ldots, A_k]$ is equal to $F[X, Y]$ for some commuting $X, Y \in F[A_1, \ldots, A_k]$. Hence $A_i \in F[X, Y]$, so we can express A_i as a polynomial $f_i(X, Y)$ where each f_i lies in the set

$$Q := \{f_i \in F[x, y] : \text{ the coefficient of } x^r y^s \text{ is zero if } r \geq n \text{ or } s \geq n\}.$$

[5] In [9], it is remarked that Sethuraman claims $n = 11$ is irreducible, but no proof has been published.

Hence the commuting k-tuple lies in the image of the polynomial function

$$\eta : \mathcal{C}(2,n) \times Q^k \to \mathcal{C}(k,n)$$
$$\eta(X,Y,f_1,\ldots,f_k) = (f_1(X,Y),\ldots,f_k(X,Y)).$$

Since the chosen k-tuple was arbitrary, the function η is surjective. The set $\mathcal{C}(2,n)$ is irreducible by Theorem 3.3.2, and Q is isomorphic to the irreducible variety $\mathbb{A}^{n^2}$, so the image $\mathcal{C}(k,n)$ is also irreducible. □

The only other variety of commuting triples that is known to be irreducible over fields of arbitrary characteristic is $\mathcal{C}(3,4)$. Much like the proof of irreducibility for $\mathcal{C}(2,n)$ relied on the usage of 1-regular matrices, we delve into 2-regularity for the purposes of this proof.

The reduction of the case where either matrix has multiple eigenvalues was not explained at source, so we filled in the details in the following proof.

Lemma 3.4.5 ([7, Theorem 8])**.** *If $A, B \in M_4(F)$ commute, then there exists a 2-regular matrix that commutes with both A and B.*

Proof. Let $(A,B) \in \mathcal{C}(2,4)$. If A has multiple eigenvalues then we can conjugate A into Weyr form and isolate each Weyr block. Since B commutes with A, the image of B under this conjugation is also block-diagonal with respect to the same partition as A by Lemma 1.2.7. Hence the problem reduces to finding a 2-regular matrix that commutes with any commuting pair of matrices of size 3 or smaller. Since all non-scalar matrices of size 3 or less are 2-regular, if either matrix is scalar then take the other, and if both are scalar then take any non-scalar matrix.

We can assume, therefore, that A and B are both non-scalar matrices with a single eigenvalue. Assume that the eigenvalues of A and B are zero, since if a 2-regular matrix commutes with the nilpotent part, then it commutes with the whole matrix.

Assume neither A nor B is 2-regular, and since neither is scalar, both have Weyr structure $(3,1)$. Hence A and B have nullity 3, so $\ker A \cap \ker B$ has dimension at least 2. Furthermore A and B both have 1-dimensional image.

Denote by $W(V)$ the image of the space V under multiplication by a matrix W. If V is 1-dimensional and $W(V) \subseteq V$, then since W has single eigenvalue 0, we must have $W(V) = 0$ i.e. $V \subseteq \ker W$. We have

$$A(\operatorname{im} B) = \operatorname{im} AB = \operatorname{im} BA = B(\operatorname{im} A) \subseteq \operatorname{im} B,$$

and since $\operatorname{im} B$ is 1-dimensional, $\operatorname{im} B$ is contained in the kernel of A. Note $B^2 = 0$ as the Weyr structure of B has 2 parts, so $\operatorname{im} B$ is also contained in the kernel of B. Therefore $\operatorname{im} B \subseteq \ker A \cap \ker B$, and a similar argument shows that $\operatorname{im} A \subseteq \ker A \cap \ker B$.

Let w_1, w_2 be linearly independent vectors contained in $\ker A \cap \ker B$ that span both $\operatorname{im} A$ and $\operatorname{im} B$. Extend this to a basis $w_1, \ldots, w_4$ of F^4 and consider the 2-regular nilpotent Weyr block

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to this basis. Multiplication by A, B, or R, sends the space spanned by $w_1, \ldots, w_4$ to the space spanned by w_1, w_2, which is killed by subsequent multiplication by A, B or R. Thus all products of A, B and R are zero, so the matrices commute. □

Let X_i be the subset of $\mathcal{C}(3,4)$ with the ith matrix 1-regular, and Y_i the subset of $\mathcal{C}(3,4)$ with the ith matrix 2-regular, with X and Y their intersections over all i. The proof of the following theorem comes in two parts: showing $\mathcal{C}(3,4) = \overline{Y}$, and then showing $\overline{Y} = \overline{X}$, and irreducibility of $\mathcal{C}(3,4)$ therefore follows from Example 3.2.9.

Theorem 3.4.6 ([7, Theorem 8]). *$\mathcal{C}(3,4)$ is irreducible.*

Proof. Let $(A, B, C) \in \mathcal{C}(3,4)$. Then by Lemma 3.4.5, there exists a 2-regular matrix R which commutes with both B and C. The line L defined to be the image of the polynomial map $t \mapsto (At + R(1-t), B, C)$ intersects Y_1 at $t = 0$, and since Y_1 is open, Y_1 is dense in L, so $(A, B, C) \in \overline{Y_1} = \overline{Y}$. Since (A, B, C) was arbitrary, $\mathcal{C}(3,4) = \overline{Y}$.

To establish $\overline{X} = \overline{Y}$, first recognise that all 1-regular matrices are also 2-regular, so $\overline{X} \subseteq \overline{Y}$. To prove the other direction, fix $(A, B, C) \in Y$. Then A commutes with some 1-regular matrix R, so $(R, 0) \in \mathcal{C}(2, A)$. Using 2-regularity of A and Theorem 3.2.11, the set $\mathcal{C}(2, A)$ is irreducible. We embed $\mathcal{C}(2, A)$ in $\mathcal{C}(3,4)$ as the image of the polynomial map

$$\begin{aligned} f : \mathcal{C}(2, A) &\to \mathcal{C}(3,4) \\ (X, Y) &\mapsto (A, X, Y). \end{aligned}$$

Then $(A, R, 0)$ lies in $X_2 \cap \operatorname{im} f$, and since $\operatorname{im} f$ irreducible, X_2 is dense in $\operatorname{im} f$, so $(A, B, C) \in \operatorname{im} f \subseteq \overline{X_2} = \overline{X}$. Therefore $Y \subseteq \overline{X}$, so $\overline{Y} = \overline{X}$ proving the theorem. □

The remaining known cases $5 \le n \le 10$ are consequences of approaching the problem via a different route. We have already seen that reducibility of commuting varieties links to the problem of bounding the dimension of a matrix algebra, and the following theorem provides yet another route for such problems to be studied. Several authors have provided techniques for perturbing triples of commuting matrices of various sizes, before using the following theorem to establish reducibility of $\mathcal{C}(3, n)$.

Theorem 3.4.7 ([22, Theorem 7.5.2])**.** *The variety $\mathcal{C}(k, n)$ over $\mathbb{C}$ is irreducible if and only if any commuting k-tuple of $n \times n$ matrices can be perturbed to simultaneously diagonalisable matrices, that is, for any k-tuple $(A_1, \ldots, A_k)$ and $\epsilon > 0$, there exist matrices $(B_1, \ldots, B_k)$ and $P \in \mathrm{GL}_n(\mathbb{C})$ such that for all i,*

- *$P^{-1}B_iP$ is diagonal, and*
- *if $c_1, \ldots, c_{n^2}$ are the entries of $(A_i - B_i)$, then $\sqrt{|c_1|^2 + \cdots + |c_{n^2}^2|} < \epsilon$.*

3.5 Dimension and Guralnick's Theorem

The final major result relating to reducibility of varieties of commuting triples which we will discuss is Guralnick's theorem.

Theorem 3.5.1 (Guralnick, [6, Theorem 3])**.** *$\mathcal{C}(3, n)$ is reducible for $n \ge 29$.*

Our final section is dedicated to a proof of Guralnick's theorem. The proof works by contradiction: our strategy is to define a property of varieties invariant under isomorphism known as *dimension*. If $\mathcal{C}(k, n)$ is irreducible, then its dimension can be expressed as a function of k and n. Given a specially chosen $n \times n$ matrix A, we look at the dimension of $\mathcal{C}(A)$. Using this, we can obtain a lower bound for $\dim \mathcal{C}(2, A)$ by looking at how many restrictions we must impose on the entries of a pair of matrices in $\mathcal{C}(A)$ to force them to commute. Then $\mathcal{C}(2, A)$ is used to generate a subset of $\mathcal{C}(3, n)$ with larger dimension than $\mathcal{C}(3, n)$: an impossibility.

The proof has developed over time, and Guralnick's original argument [6, Theorem 3] works only for $n \ge 32$. The matrix A used is displayed below, as well as a certain form of matrix in the centraliser of A used by Guralnick. All blocks are of size $s \times s$, so $n = 4s$.

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & A & B & C \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then s^2 commutation restrictions are required, giving $\dim C(2, A) \geq 7s^2$.

An adaptation of the proof found in [11] extends the argument to $n = 30, 31$. Given positive integers $a + 3b = n$, the authors instead consider the matrix A equal to the direct sum of an $a \times a$ zero block and b 3×3 nilpotent Jordan blocks. This is a Jordan matrix of Jordan structure

$$(\overbrace{3, \ldots, 3}^{b \text{ times}}, \overbrace{1, \ldots, 1}^{a \text{ times}})$$

which by Jordan-Weyr duality has Weyr structure $(b + a, b, b)$, the Weyr structure of the matrix A we will consider. A small adjustment in [22] covers $n = 29$ as well, the trick being that the subset generated by $\mathcal{C}(2, A)$ obtained in [11] is contained in a variety strictly contained in $\mathcal{C}(3, n)$, so must have strictly smaller dimension.

We opt for a simplified definition of dimension provided by [22, Proposition 7.8.3].

Definition 3.5.2. Let $V \subseteq \mathbb{A}^n$ be an irreducible variety. We define the *dimension* of V to the the largest m such that for some distinct indeterminates $x_{i_1}, \ldots, x_{i_m} \in F[x_1, \ldots, x_n]$, the only polynomial $f \in F[x_{i_1}, \ldots, x_{i_m}] \subseteq F[x_1, \ldots, x_n]$ is the zero polynomial.

The concept can be extended to arbitrary sets in affine n-space.

Definition 3.5.3 ([22, p355])**.** Let $X \subseteq \mathbb{A}^n$. If X is a variety, define $\dim X$ to be the maximum dimension of its irreducible components. Otherwise, define $\dim X$ to be the dimension of $\overline{X}$.

We state some results on dimension which are used in the proof of Guralnick's theorem. Proofs can be found in [22, Chapter 7].

Lemma 3.5.4.

- *Dimension is invariant under isomorphism.*
- *If* $V \subseteq W \subseteq \mathbb{A}^n$ *then* $\dim V \leq \dim W$*.*
- *If* $V \subseteq W \subseteq \mathbb{A}^n$ *are varieties with* W *irreducible then* $\dim V = \dim W$ *implies* $V = W$*.*
- *For* $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$*, we have* $\dim(V \times W) = \dim V + \dim W$*.*

The first substantial step towards proving Guralnick's theorem is determining the dimension of $\mathcal{C}(k, n)$ when the variety is irreducible.

Proposition 3.5.5 ([22, Lemma 7.9.1])**.** *If $\mathcal{C}(k,n)$ is irreducible then its dimension is $n^2+(k-1)n$.*

Proof. The set U of 1-regular matrices is open in $M_n(F)$, so its closure has dimension n^2. Therefore by definition, U has dimension n^2. The subset X_1 of matrices $(A_1,\dots,A_k)$ with A_1 1-regular is open, and therefore dense by irreducibility of $\mathcal{C}(k,n)$, so its dimension is equal to the dimension of $\mathcal{C}(k,n)$. Recall from Theorem 2.2.4 that the matrices commuting with A are exactly those of the form $p(A)$ where p has degree at most $(n-1)$. Then the polynomial map

$$f : U \times P_n^{k-1} \to X_1,$$
$$(A, p_1, \dots, p_{k-1}) \mapsto (A, p_1(A), \dots, p_{k-1}(A)),$$

is an isomorphism, and $P_n \cong \mathbb{A}^n$ so $\dim \mathcal{C}(k,n) = \dim X_1 = n^2+(k-1)n$. $\square$

The next step is to construct a subset of $\mathcal{C}(3,n)$ with dimension exceeding this. We utilise a family of such subsets which depend on a pair $a,b \in \mathbb{N}$.

Consider the nilpotent Weyr block W with Weyr structure $(b+a,b,b)$, and denote by Γ_0 the set of pairs (K,K') of matrices of the form

$$K = \begin{pmatrix} 0 & A & B & C \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K' = \begin{pmatrix} 0 & A' & B' & C' \\ 0 & 0 & 0 & D' \\ 0 & 0 & 0 & B' \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where we have blocked the matrices according to the partition[6] (b,a,b,b).

The set Γ_0 is isomorphic to $\mathbb{A}^{4ab+4b^2}$. Note that K and K' both commute with W but do not necessarily commute with each other. Comparing the entries of the products KK' and $K'K$ gives us the commutation requirement

$$AD' + BB' = A'D + B'B \in M_b(F),$$

giving a total of b^2 equations to impose on Γ_0 to obtain the set $\Gamma = \Gamma_0 \cap \mathcal{C}(2,W)$. To find the dimension of Γ, we use the following theorem.

Lemma 3.5.6 ([22, Theorem 7.8.6])**.** *Let $V \subseteq \mathbb{A}^n$ over F and $f \in F[x_1,\dots,x_n]$. If $V \cap V(f) \neq \emptyset$ but V is not contained in $V(f)$ then*

$$\dim(V \cap V(f)) = \dim V - 1.$$

[6]In this section, we temporarily violate our requirement that a partition $(n_1,\dots,n_s)$ of n must satisfy $n_1 \geq n_2 \geq \cdots \geq n_s$.

The main consequence of this result is that imposing a single polynomial condition on a variety can reduce its dimension by at most 1, so imposing r further polynomial conditions on a variety drops the dimension by at most r (provided $V \cap V(f_1) \cap V(f_2) \cap \cdots \cap V(f_r)$ is nonempty). Therefore we obtain the inequality

$$\dim \Gamma \geq (4ab + 4b^2) - b^2 = 4ab + 3b^2.$$

The next step is to "embed" Γ into $\mathcal{C}(3, n)$. Conjugation preserves commutativity, as does adding scalar matrices, and exploiting these facts allows us to maximise the dimension of the "embedded" version of Γ in $\mathcal{C}(3, n)$. To obtain a bound on the dimension of this set, we use the dimension of fibres theorem.

Theorem 3.5.7 (Dimension of Fibres Theorem, [22, Theorem 7.8.12])**.** *Let $f : V \to \mathbb{A}^m$ be a polynomial map, where $V \subseteq \mathbb{A}^n$. Then there exists $w \in \operatorname{im} f$ such that*

$$\dim f^{-1}(w) \geq \dim V - \dim \operatorname{im} f.$$

Lemma 3.5.8 ([22, Lemma 7.9.2])**.** *Consider the polynomial map*

$$f : \Gamma \times \mathbb{A}^3 \times \mathrm{SL}_n(F) \to \mathcal{C}(3, n)$$
$$(K, K', \lambda_1, \lambda_2, \lambda_3, P) \mapsto (P^{-1}WP + \lambda_1 I, P^{-1}KP + \lambda_2 I, P^{-1}K'P + \lambda_3 I).$$

We have $\dim \operatorname{im} f \geq n^2 + 2ab - a^2 + 3$.

Proof. We first find the dimension of a fibre $f^{-1}(A, B, C)$ with $(A, B, C) \in \operatorname{im} f$. Therefore

$$(A, B, C) = (P^{-1}WP + \lambda_1 I, P^{-1}KP + \lambda_2 I, P^{-1}K'P + \lambda_3 I)$$

for some $(K, K') \in \Gamma, (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{A}^3, P \in \mathrm{SL}_n(F)$. Consider some

$$\gamma := ((L, L'), (\mu_1, \mu_2, \mu_3), Q) \in \Gamma \times \mathbb{A}^3 \times \mathrm{SL}_n(F).$$

Then $\gamma \in f^{-1}(A, B, C)$ if and only if the following criteria hold:

1. $P^{-1}WP + \lambda_1 I = Q^{-1}WQ + \mu_1 I$, so equating eigenvalues gives us $\lambda_1 = \mu_1$, and we therefore obtain $QP^{-1}W = WQP^{-1}$, so $QP^{-1} \in \mathcal{C}(W)$.
2. $P^{-1}KP + \lambda_2 I = Q^{-1}LQ + \mu_2 I$, so $\lambda_2 = \mu_2$, and $L = QP^{-1}K(QP^{-1})^{-1}$.
3. Similarly, $\lambda_3 = \mu_3$ and $L' = QP^{-1}K'(QP^{-1})^{-1}$.

Thinking of Z as QP^{-1}, we conclude that

$$\gamma = ((ZKZ^{-1}, ZK'Z^{-1}), (\lambda_1, \lambda_2, \lambda_3), ZP),$$

where any choice of $Z \in \mathcal{C}(W) \cap \mathrm{SL}_n(F)$ determines a unique γ in the preimage of (A, B, C). That is, we have the following isomorphism of varieties:

$$\begin{aligned} \mathcal{C}(W) \cap \mathrm{SL}_n(F) &\to f^{-1}(A, B, C), \\ Z &\mapsto ((ZKZ^{-1}, ZK'Z^{-1}), (\lambda_1, \lambda_2, \lambda_3), ZP), \end{aligned}$$

its inverse being the polynomial map $(L, L', \mu_1, \mu_2, \mu_3, Q) \mapsto QP^{-1}$. It follows that $\dim f^{-1}(A, B, C) = \dim(\mathcal{C}(W) \cap \mathrm{SL}_n(F))$. From Remark 1.2.6, $\mathcal{C}(W) \cong \mathbb{A}^{(b+a)^2+b^2+b^2}$, so $\dim \mathcal{C}(W) = a^2 + 2ab + 3b^2$. Since $\mathrm{SL}_n(F)$ is determined by the vanishing of the single polynomial $(\det -1)$, intersection with $\mathrm{SL}_n(F)$ drops the dimension by 1, so $\dim f^{-1}(A, B, C) = a^2 + 2ab + 3b^2 - 1$.

Using the dimension of fibres theorem then gives us

$$\begin{aligned} \dim \operatorname{im} f &\geq \dim(\Gamma \times \mathbb{A}^3 \times \mathrm{SL}_n(F)) - \dim f^{-1}(A, B, C) \\ &= \dim \Gamma + \dim \mathbb{A}^3 + \dim \mathrm{SL}_n(F) - \dim f^{-1}(A, B, C) \\ &\geq (4ab + 3b^2) + 3 + n^2 - 1 - a^2 - 2ab - 3b^2 + 1 \\ &= n^2 + 2ab - a^2 + 3. \end{aligned}$$

□

Proof (of Guralnick's Theorem). Assume that $\mathcal{C}(3, n)$ is irreducible. Let f be the function from the previous theorem. Note that every triple in $\operatorname{im} f$ contains a matrix with a repeated eigenvalue. The set X of such matrices is closed, since any symmetric polynomial in the eigenvalues of a matrix can be expressed as a polynomial in the entries of the matrix (see [22, Proposition 7.1.10] for a proof of this). Note that $X \neq \mathcal{C}(3, n)$; take the triple (D, D, D) with D a diagonal matrix with each diagonal entry distinct.

Since $\operatorname{im} f \subseteq X \subset \mathcal{C}(3, n)$, then if $\mathcal{C}(3, n)$ is irreducible, we have

$$\begin{aligned} n^2 + 2ab - a^2 + 3 &= \dim \operatorname{im} f \\ &\leq \dim X \\ &< \dim \mathcal{C}(3, n) \\ &= n^2 + 2n \\ &= n^2 + 2a + 6b \end{aligned}$$

Therefore if for a given n there exist $a, b \in \mathbb{N}$ such that $a + 3b = n$ and

$$0 < a^2 + (2 - 2b)a + (6b - 3)$$

then we derive a contradiction, so if we consider $a^2 + (2 - 2b)a + (6b - 3)$ to be a polynomial in a, we need the polynomial to have real roots, meaning $b^2 - 8b + 4 \geq 0$, so $b \geq 8$, and we need a to lie in the interval between them. That is, we can take a to be any positive integer satisfying

$$(b-1) - \sqrt{b^2 - 8b + 4} \leq a \leq (b-1) + \sqrt{b^2 - 8b + 4}.$$

Plugging in $a = 5, 6, 7, 8$ with $b = 8$ gives us the required contradiction for matrices of size 29, 30, 31 and 32. To prove the theorem for all larger matrix sizes, observe that the size of the interval which must contain a increases as we increment b. If $n = a + 3b$ with

$$(b-1) - \sqrt{b^2 - 8b + 4} \leq a \leq (b-1) + \sqrt{b^2 - 8b + 4},$$

and $a + 1$ does not lie in this interval, then since each interval has size at least 4, we know that the integers $(a-3), (a-2), (a-1)$ and a are contained in the interval. Then we can consider the interval

$$b - \sqrt{(b+1)^2 - 8(b+1) + 4} \leq a - 2 \leq b + \sqrt{(b+1)^2 - 8(b+1) + 4},$$

noting that if $a - 2$ being contained in this interval is equivalent to having

$$(b-1) - \sqrt{(b+1)^2 - 8(b+1) + 4} \leq a - 3 \leq (b-1) + \sqrt{(b+1)^2 - 8(b+1) + 4},$$

which we know to be true. We have $n + 1 = (a-2) + 3(b+1) = (a+1) + 3b$, and at least one of these eventualities is covered. $\square$

The results of this chapter are summarised in the diagram below.

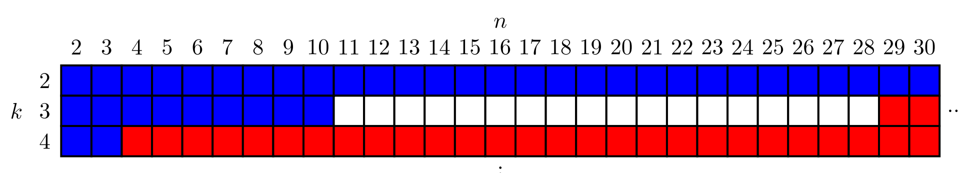

Figure 3.2: Diagram representing the current state of the problem, where blue boxes represent reducible varieties, red boxes represent reducible varieties, and white boxes represent cases where the problem is still open.

Chapter 4

Conclusion

To summarise, in Chapter 1 we established Weyr matrices as a canonical form, and covered their advantageous properties such as the form of their centraliser and the block shifting effect. Chapter 2 provided an introduction to the theory of matrix algebras and examined counterexamples which violate certain bounds on their dimensions. An insight into the computational strategy aiming to quash Gerstenhaber's problem was also provided. Results from algebraic geometry required to interface with dimension of matrix algebra problems were developed in Chapter 3, together with an examination of the current knowledge on the (ir)reducibility of $\mathcal{C}(k, n)$.

Throughout, we have seen examples of using Weyr matrices to improve existing techniques, such as in the proof of Guralnick's theorem, and to open up a new avenue of attack for Gerstenhaber's problem. The little-known canonical form has been used to give alternative proofs of known results, such as the equivalence of definitions of k-regularity (Lemma 3.1.7).

In terms of directions for further study, there is certainly potential to advance the computing strategy outlined in Chapter 2. The authors of [10] remark that they are 'not experts in computing', so ideas for improvements to the computational approach may come relatively easily to a skilled programmer. A specific direction in this realm which comes to mind is the automation of the selection of the parameters determining the frequency of nonzero entries of generated commuting triples[1]. On the algebraic geometry side, a paper of Ngo and Šivic [21] connects reducibility of $\mathcal{C}(3, n)$ to reducibility of varieties of tuples of commuting nilpotent matrices, which may prove a promising approach to closing the current (ir)reducibility gap.

[1]In an email, Kevin O'Meara remarked that the selection of such parameters requires skill 'bordering on that of a concert pianist'!

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