University of Sheffield

UNDERGRADUATE RESEARCH INTERNSHIP, SUMMER 2024

1. The UGRI scheme

The UnderGraduate Research Internship (UGRI) scheme is an internal programme of the School of Mathematics and Statistics offering undergraduate students the opportunity to participate in a intensive 4-week research project or directed reading under the supervision of a PhD student or a postdoctoral fellow, possibly with input from members of the academic staff.

You will be able to get a first taste of what research in Mathematics (Pure or Applied) and Probability and Statistics is like, and it is possible that your work will lead to a publication in an academic journal. Even if that doesn't happen, this research experience can make an excellent addition to your CV and can inform your decisions regarding postgraduate studies.

Our department is devoted to creating an inclusive and welcoming environment to all those who have a passion for Mathematics and/or Statistics. In light of this, the UGRI programme was set up with the primary goal of offering research experience to students who are *typically underrepresented in the mathematical sciences* at research level. These *underrepresented groups include, but are not limited to: people from low-income backgrounds; women; people with a disability; people who identify as LGBTQ+; and people in BAME communities.*

Eligibility for UGRI is limited to students in their 1st, 2nd, and 3rd year To apply, you need to fill in the application form; the link for this is in the email but also here (click me, I am a link) for convenience. You can apply for up to two projects but, if successful, you can only work on one. The deadline for applications is **5pm on the 22nd of March**. Those shortlisted will be notified most likely after the Easter holiday, and interviews will be scheduled at a later point.

If you have any questions about the programme, your eligibility, or the application process, you should direct them to Dr Dimitrios Roxanas.

2. Projects

Project 1
Title: Exploring Homological Algebra: The Universal Coefficient Theorem
Supervisor: Jacob Saunders
Area: Pure Mathematics (Algebra/Topology)

Description: Homology and cohomology theories are used throughout many areas of pure maths to distil information from one context into information about abelian groups, which is typically easier to understand and work with. The branch of maths which deals with the resulting groups is called homological algebra.

The student will, over the course of this project, develop an understanding of the core concepts of homological algebra. These include the notion of a chain complex, homology and cohomology, and exact sequences. The eventual goal of the project is to understand the proof of the universal coefficient theorem: a result relating the homology and cohomology of a chain complex.

Such knowledge is essential for any student considering studying topology, and useful in general for a student who is thinking of pursuing pure maths.

Prerequisites: The module MAS2006 Group Theory is a prerequisite.

Milestones: Milestones will include:

- Understanding the definitions of homology, chain complex and exact sequence.
- Learning about "diagram-chasing". This is a method of proof which involves pushing group elements around diagrams formed of groups and homomorphisms. Diagram chasing can be used to prove injectivity and surjectivity of a homomorphism. The student will learn to apply this to prove results such as the "five lemma" and the "snake lemma".
- Understanding why taking groups of homomorphisms ("hom" groups) is "left exact", and how the exact sequence is extended using "ext" groups.
- The proof of the universal coefficient theorem itself.

References that the student may find useful are:

- Weibel - Introduction to Homological Algebra, Chapter I (which gives the basic definitions e.g. chain complex, homology, as well as a number of exercises)
- Switzer - Algebraic Topology, Chapter 13 (which contains a thorough proof of the universal coefficient theorem)

Project 2
Title: A First Taste of the Trace Formula
Supervisor: Ho Leung Fong
Area: Pure Mathematics (Algebra)

Description: The Langlands program is arguably the most important problem in modern number theory. One key tool for attacking it is the trace formula. It calculates the trace of an operator acting on a space in 2 ways, which enables one to get 2 expressions for the trace. Typically, one tries to deduce properties for the spectral side (more mysterious) by analysing the geometric side (more accessible). Some applications are constructions of Galois representations, the Jacquet-Langlands correspondence, proof of local Langlands correspondence, etc. Among other things, these are used in the proof of the modularity theorem, which is the major ingredient in the proof of Fermat's last theorem. Although the trace formula was discovered at least 50 years ago, it is still actively researched by mathematicians. In fact, Ngô got the Fields medal in 2010 for proving the fundamental lemma, which is an important component in the trace formula.

This project aims to understand some simpler versions of the trace formula and see some applications of them. The main reference will be the note by David Whitehouse on `https://www.ma.imperial.ac.uk/~buzzard/MSRI/Whitehouse_tf.pdf`.

Milestones:

- For the 1st week, the student will learn some basic concepts of group representations, such as direct sums of representations, irreducibility of representations, inductions, and some examples.
- For the 2nd and 3rd week, the student will learn the proof of the trace formula for finite groups. As an application, the student can deduce Frobenius reciprocity from the trace formula.
- In the 4th week, the student will try to understand the statement of the trace formula for compact quotients in various special cases and how the proof will be modified. As an application, one can deduce the Poisson summation formula from the trace formula.

Prerequisites: Students should be taking, or have completed, MAS107 'Foundations of Pure Mathematics' (formerly MAS114) and MAS2006 'Group Theory'.

Project 3
Title: Supersymmetric Quantum Mechanics
Supervisor: Dr Fabrizio Del Monte
Area: Applied Mathematics (Mathematical Physics)

Description: Quantum mechanics provides the fundamental description of our world, but the list of systems whose solution is exactly known is extremely limited. Only recently people have been starting to realise that a common feature of these systems is the presence of a hidden supersymmetry, which has provided a unified viewpoint on exactly solvable quantum mechanical models. Exploiting this symmetry gives a new perspective on classic results in Quantum Mecha-nics: so-called Supersymmetric Quantum Mechanics (SUSYQM) is still being developed today as an important set of tools in fields as diverse as String Theory, Geometry, Representation Theory to name a few.

In this project, we will first review the traditional solution of the Schrödinger equation for the harmonic oscillator and hydrogen atom using orthogonal polynomials. We will see how these results can be rephrased within the formalism of Supersymmetric Quantum Mechanics, which we will use to give them a new derivation. This will shed light on the general properties of systems that can be solved exactly in Quantum Mechanics.

If time permits, the student can choose between one of two advanced topics: the solution of the hydrogen atom fine structure using SUSYQM, or the Witten Index in geometry.

Prerequisites At the very least, the student should be familiar with Ordinary Differential Equations (MAS2002), but some basic familiarity with Quantum Mechanics and the Schrödinger equation (MAS324) is not necessary but advisable.

References:

- Supersymmetric Quantum Mechanics: An Introduction (2nd ed.) by A. Gangopadhyaya, J. Mallow, C. Rasinariu
- Lectures on Supersymmetric Quantum Mechanics by David Tong

Project 4
Title: Predators and Preys
Supervisor: Alessandro Monteverdi
Area: Applied Mathematics (Dynamical systems)

Description: Differential equations stand as fundamental concepts in the realms of physics and mathematics, offering a powerful tool to depict and anticipate real-world phenomena. We will delve into the exciting world of non-linear differential equations and learn about the different types of attractors and how they can be classified, as well as the concept of isocline, which is a graphical representation of the equilibrium points of the system.

This theoretical foundation will be applied to a concrete scenario, attempting to predict predator and prey populations (Lotka-Volterra equations) depending on internal and external factors (such as predation rate, birth rate, etc.). Then we will focus on the stability analysis of the system, including the local and global stability of the equilibrium points and the conditions for stability.

We will also learn about the Poincaré-Bendixson theorem and its application to the predator-prey system. By using this theorem, we can figure out if the populations of predators and prey will settle into a steady pattern or keep oscillating and depending on the students' interests, we can either prove the theorem or simply use the result in the analysis.

Finally, we will have the opportunity to work with the predator-prey system numerically and explore its dynamics by using numerical techniques. We will use the knowledge we have gained to analyse and predict the behaviour of the system and understand how changes in the parameters affect the populations of predators and prey.

In this project, we will primarily focus on understanding the predator-prey system. However, once we have a good grasp of this system, we could also introduce some of the theory behind the Lorenz system. This is a more complex model of non-linear differential equations that is known for its connection to the butterfly effect. It's used to simulate changes in weather patterns and can exhibit chaotic behaviour. While we won't go into deep details about the Lorenz system, we will mention it as an example of a more complex system and its possible implications on the real world.

Milestones:

(1) A brief introduction to non-linear differential equations
(2) Introduction to the concept of attractors, equilibrium points, and isoclines.
(3) Study of preditor-prey system (Lotka-Volterra equations).
(4) Poincaré-Bendixson theorem and numerical analysis.
(5) Introduction to Lorenz system (optional) and conclusions.

Prerequisites: Prior knowledge of non-linear differential equations is not essential for this project, as we will not be delving directly into the mathematical formalism. Familiarity with programming languages, specifically Mathematica, can be advantageous for numerical studies, but is not a requirement.

References:

(1) Wikipedia can be useful: `https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations`
(2) M. Hirsch, Differential Equations, Dynamical Systems and an Introduction to Chaos, Academic Press, 2012 (Chapter 11).
(3) G. Teschl, Ordinary Differential Equations and Dynamical Systems, American Mathematical Society, 2004 (Chapter 8).
(4) A. Fasano, S. Marmi, Analytical Mechanics, OUP Oxford, 2006 (Mostly for reference and general use).

These textbooks cover a wide range of topics and provide a solid mathematical foundation for the subject. Although they may not be the most suitable primary textbooks for a project focused on non- linear differential equations, they can be a useful resource to consult for furthering your understanding of certain aspects of the predator-prey system.

Project 5
Title: Spinning Around Black Holes
Supervisor: Ethan German
Area: Applied Mathematics (Mathematical Physics)

Description: In 1915, Einstein published his theory of general relativity (GR). In this theory, he proposed that matter curves spacetime, and that the curvature of spacetime dictates how matter itself moves. Two underlying equations describe this relationship between matter and curvature: the Einstein field equations, which, given a configuration of matter, tell you how your spacetime curves; and the geodesic equation, which, given a spacetime curvature, tells you how your particles move in that spacetime.

Soon after GR's publication, Karl Schwarzschild published his solution to the Einstein Field equations, giving us the first black hole spacetime, known as the Schwarzschild spacetime. In this project we will study orbits of particles in the Schwarzschild spacetime by numerically solving the geodesic equation. After this we will look at how those orbits change if the particle can spin on its own axis.

A rough outline of the project will be:

- A brief introduction to GR and The Schwarzschild Solution
- Use Wolfram Mathematica to calculate geodesics for test particles in the Schwarzschild spacetime.
- Understand the equations of motion for a test-particle that has some spin in Schwarzschild solution.
- Use Wolfram Mathematica to numerically integrate the equations of motion to obtain trajectories of spinning particles.

Prerequisites: Required: MAS2002, MAS2005, Preferred: MAS314.
Programming knowledge in any language is useful, however we will be using Wolfram Mathematica.

References: (reading these in depth before the project starts is optional)

- Spacetime and Geometry, Sean Carroll (2004), Cambridge University Press
  - This is a textbook about General Relativity, it contains a nice chapter about the Schwarzschild Solution and its geodesics.
- A spinning test body in the strong field of a Schwarzschild black hole, Theocharis A Apostolatos (1996) DOI 10.1088/0264-9381/13/5/005
  - This is a paper about how we can model the trajectory of particles that spin around Schwarzschild black hole.

Project 6
Title: Field Theory: A Primer
Supervisor: Abhinove Seenivasan
Area: Applied Mathematics (Mathematical Physics)

Description: Fields are a central notion in modern theoretical physics, and all the fundamental theories of nature are formulated in the language of field theories. Broadly, a field theory may be classical (such as General Relativity) or quantum (such as Quantum Electrodynamics) and there are interesting connections between the two. This project aims to introduce the concept of a field theory and explore how it differs from a particle viewpoint of physics, while emphasising the role of symmetry and conservation laws in both approaches.

Milestones:

- Reviewing Lagrangian and Hamiltonian methods for classical dynamics
- Introducing special relativity and index notation
- Learning to work with simple classical field theories such as scalar field theory
- Noether's theorem
- Rewrite well known classical physics (for e.g., Maxwell's equations) in the language of field theory.

Possible Further Directions:

- Classical gauge fields
- Gravity as a field theory
- Introduction to quantum field theory
- Symmetries in a field theory
- The geometry behind field theories

Prerequisites: MAS2001/MAS211 would be a good prerequisite for interested students.
References: It would be useful to look at these references (in this order) ahead of the project. [1] and [2] provide a brief introduction to field theory without a lot of technical details or mathematics. Particularly, [2] gives an overview of field theory by starting with Lagrangian dynamics for particle systems (in 20 mins!). References [3] and [4] are more pedagogical and detailed.

(1) Wikipedia contributors, "Field (physics) — Wikipedia, the free encyclopedia," `https://en.wikipedia.org/w/index.php?title=Field_(physics)&oldid=1206436799`.
(2) Physics with Elliot, "Field theory fundamentals in 20 mins," `https://www.youtube.com/watch?v=13hCkUiu_mI`.
(3) D. Tong, "Electromagnetism and relativity." [Online]. Available: `https://www.damtp.cam.ac.uk/user/tong/em/el4.pdf`.
(4) ——, "Classical field theory." [Online]. Available: `https://www.damtp.cam.ac.uk/user/tong/qft/one.pdf`.

Project 7
Title: Science Fiction in Free Fall: Motion in General Relativity
Supervisor: Dr Sebastian Schuster
Area: Applied Mathematics (Mathematical Physics)

Description: This project will study geodesic motion in curved space-times—put differently, free fall in general relativity, the current theory of gravitation. In particular, we will look at geodesic motion in so-called tractor beam space-times [1]. Tractor beam space-times are hypothetical arrangements of "exotic matter" whose pressure can push and pull particles as desired, by appropriately placing this matter. Like the related and more famous warp drive space-times—arrangements of matter meant to gravitationally move a small patch along a chosen path—they are "reverse engineered". Normally, one solves general relativity by integrating its partial differential equation, the Einstein equation, for a fixed source (the stress energy tensor) to find the metric describing the space-time. A reverse-engineered metric instead uses the metric as input to calculate the stress-energy tensor (the matter content of the space-time) from the Einstein equation through differentiation.

Two sets of related questions arise quite naturally in this context: For one, we might want to ask whether a given metric tensor/space-time and its source term/stress-energy tensor are "plausible" or "physical". For another, we would want to check whether a particle encountering a tractor beam will behave as the motivation behind the space-time promised. If this is not the case, this would give an indication that the original goal—a tractor beam—has not been fully achieved yet, a mild version of a concern about physical and easy-to-interpret implications of a metric. The issue of how easy such a criterion for plausibility is to interpret is rather deep: often, one encounters the already mentioned notion of "exotic matter". This is a short-hand for matter that seems unphysical/implausible, because of a vague(!) notion of "negative mass", encoded in so-called energy conditions [3]. Warp drives and tractor beams both require exotic matter [1,4], but so far faster-than-light warp drives have by far the most obvious plausibility issues.

Milestones: Depending on the student's strengths and interests, the project can be developed in the direction of philosophy of science, numerical solution of ordinary differential equations, or general relativity. The project can easily be scaled from understanding existing literature to genuine new research results. The first week will be used to become familiar with the notation and core concepts encountered in the relevant literature. This and the student's interests will then determine the nature of the precise task done in the second and third week: numerical or analytical integration of geodesic equations, exploratory calculations with energy conditions, or the more literature-heavy and discursive philosophical aspects. These choices would influence which direction a potential, original research output would take. One such output would be to find tractor beam space-times motivated not by pressure terms in the stress-energy but rather by the motion predicted by the geodesic equation. The final week will be left for preparing the presentation of the project's work.

Prerequisites: (read "/" as "or", not "and")

- Calculus/Analysis/Differential Equations: MAS2001 (Mathematics Core II)/ MAS2002 (Differential Equations)/ MAS2004 (Analysis and Algebra)/ MAS2005 (Vector Calculus and Dynamics)/ MAS211 (Advanced Calculus and Linear Algebra)/ MAS280 (Mechanics and Fluids)
- Linear Algebra: MAS2004 (Analysis and Algebra)/ MAS211 (Advanced Calculus and Linear Algebra)
- Familiarity with MAS254 (Computational and Numerical Methods)/ MAS212 (Scientific Computing and Simulation) can prove beneficial.

References:

(1) J. Santiago, S. Schuster, M. Visser, "Tractor beams, pressor beams, and stressor beams in general relativity", arXiv:2106.05002, Universe 7(8), 271 (2021), DOI:10.3390/universe7080271.
(2) S. Schuster, "Frenemies with Physicality: Manufacturing Manifold Metrics", arXiv:2305.08725, Essay written for the Gravity Research Foundation 2023 Awards for Essays on Gravitation.
(3) E. Curiel, "A Primer on Energy Conditions", arXiv:1405.0403, In: "Towards a Theory of Spacetime Theories", p.43—104, Eds.: D. Lehmkuhl, G. Schiemann, E. Scholz, Springer (2017), ISBN:978-1-4939-3209-2, DOI:10.1007/978-1-4939-3210-8_3.
(4) J. Santiago, S. Schuster, M. Visser, "Generic warp drives violate the null energy condition", arXiv:2105.03079, Phys.Rev.D 105(6), 064038 (2022), DOI:10.1103/PhysRevD.105.064038.

Project 8
Title: A Chart of Glass: Mimicking Black Holes with Dielectric Media
Supervisor: Dr Sebastian Schuster
Area: Applied Mathematics (Mathematical Physics)

Description: Many predictions of general relativity, the current theory of gravitation, and its objects, four-dimensional space-times, have a tendency to be hard to visualize or access experimentally. To help with this, analogue space-times were conceived. They try to mimic aspects of general relativity in other (more table-top-friendly) fields of physics. Relativity, space-times and the wave equation are built on the finite speed of light in vacuum. In an analogue space-time, a similar role is played by, for example, the finite speed of surface waves in water or the finite speed of light in dielectric media (insulators that can be polarized in an electric field, like glass). This latter idea, electromagnetic media, is now over one-hundred years old, and was introduced by Gordon [1]. Usually, such analogue space-times are derived from wave equations in a given theory of physics, and then forming an analogy between different fields' wave equations. However, this oldest idea of an analogue space-time turns out to be more closely related to an analogy in "matrix multiplication" (tensor algebra) than an analogy in partial differential equations [2,3]. As an often overlooked consequence of this, light propagation in the dielectric mimicking a space-time will behave subtly and slightly different from light propagation in the mimicked, astrophysical space-time [4]. This is closely related to how geodesics on the surface of the Earth, great circles, generically appear curved on a flat map of the Earth. Much work on such cartographic distortions in analogues has focussed on cosmological space-times [4]. This project will try to apply this to mimicked black holes.

Milestones: The first week will be spent getting familiar with the required notation and literature. Besides some tensor calculus, this will include simple differential geometry and mathematical cartography. Depending on the interests of the student, the middle two weeks of the project can then be spent on visualizing the existing work by Fathi and Thompson [4], contrasting it with traditional two-dimensional cartography on the Earth ("geodesy"), or analytically quantifying these effects using the so-called "non-metricity tensor" [3]. An application of these ideas to analogue black hole space-times instead of analogue cosmological space-times could become an original research output in the latter stages of the project. The final week will be left for preparing the presentation of the project's work.

Prerequisites: (read "/" as "or", not "and")

- Calculus/Analysis/Differential Equations: MAS2001 (Mathematics Core II)/ MAS2002 (Differential Equations)/ MAS2004 (Analysis and Algebra)/ MAS2005 (Vector Calculus and Dynamics)/ MAS211 (Advanced Calculus and Linear Algebra)/ MAS280 (Mechanics and Fluids)

- Linear Algebra: MAS2004 (Analysis and Algebra)/ MAS211 (Advanced Calculus and Linear Algebra)
- Familiarity with MAS254 (Computational and Numerical Methods)/ MAS212 (Scientific Computing and Simulation) can prove beneficial.

References:

(1) W. Gordon, "Zur Lichtfortpflanzung nach der Relativitätstheorie", Ann. Phys. 377(22), p.421—456 (1923), DOI:10.1002/andp.19233772202, English translation available at `https://www.neo-classical-physics.info/uploads/3/4/3/6/34363841/gordon_-_optical_metrics.pdf`
(2) S. Schuster, M. Visser, "Effective metrics and a fully covariant description of constitutive tensors in electrodynamics", arXiv:1706.06280, Phys.Rev.D 96(12), p.124019 (2017), DOI:10.1103/PhysRevD.96.124019
(3) S. Schuster, M. Visser, "Electromagnetic analogue space-times, analytically and algebraically", arXiv:1808.07987, Class.Quant.Grav. 36(13), p.134004 (2018), DOI:10.1088/1361-6382/ab2159
(4) M. Fathi and R.T. Thompson, "Cartographic distortions make dielectric spacetime analog models imperfect mimickers", arXiv:1602.08341, Phys.Rev.D 93(12), p.124026 (2016), DOI:10.1103/PhysRevD.93.124026

Project 9
Title: Survival Analysis Methods with Non-Proportional Hazards
Supervisor: James Salsbury
Area: Statistics (Medical Statistics)

Description: Randomised Controlled Trials (RCTs) are widely considered the gold standard method for assessing the efficacy of a new drug. The design of RCTs requires careful consideration to give the best possible chance of observing a successful trial. In this design process, a power value is selected, typically set at 80% or 90%. This power represents the probability of detecting a difference if one truly exists. However, achieving the desired power depends on accurately specifying parameters such as sample size and the chosen method of analysis. If these are mis-specified then we run the risk of underpowering the trial, thus reducing the chances of observing a successful trial. This issue is particularly problematic for companies investing significant time and resources into conducting these RCTs.

In a typical survival (time-to-event) RCT, it is commonly assumed that events (deaths etc) occur at a constant rate throughout the trial, a concept known as proportional hazards. The log-rank test, frequently used for survival analysis, is most effective under this assumption. However, there is a growing number of observed trials where this assumption is violated.

The Fleming–Harrington class of hypothesis tests encompasses weighted log-rank tests, offering a solution to scenarios where proportional hazards do not hold. By incorporating different parameters that allow for the weighting of times, these tests provide flexibility in their application, addressing the limitations of the standard log-rank test.

This project seeks to conduct a series of simulation experiments to systematically compare the statistical power of the log-rank test and weighted log-rank tests in the presence or absence of proportional hazards. Through simulations, we intend to evaluate and quantify the performance of these methods under various conditions, providing insights into their comparative strengths and weaknesses.

Milestones:

- Background reading on RCTs, survival trials, non-proportional hazards and (weighted) log-rank tests
- Develop a simulation framework in R to model non-proportional hazards in survival trials.
- Generate plots and tables comparing log-rank and weighted log-rank tests under different scenarios

Pre-requisites:

- A strong knowledge of R would be very useful.
- MAS2010 (MAS223) Statistical Inference and Modelling (Essential)
- MAS361 Medical Statistics (Useful)

Project 10
Title: Understanding Animal Behaviour Through Statistical Movement Analysis
Supervisor: Dominic Grainger
Area: Statistics (Statistical Ecology)

Description: Statistical movement ecology is a rapidly growing (and very important) research area. Exploring how and why animals move is crucial for aiding conservation efforts and assessing the impact of anthropogenic climate change. In this project, using GPS tracking data, we will model animal movement as a Markov process in order to glean information about both an animal's movement dynamics and behaviour.

Initially, you will familiarise yourself with relevant literature on commonly used methods for animal movement modelling. Specifically, we'll be focusing on the Hidden Markov model (HMM). By the end of the first week, you'll be able to answer important questions such as: 'Why is the HMM so widely used by ecologists?', 'What key assumptions does the HMM make?', or 'When is it not appropriate to use the HMM?'.

From here you can have more independence if you wish. You may prefer to focus on the theoretical/ inference side of modelling or work directly with data in a more computationally intensive way, using R packages such as moveHMM or momentuHMM. We may get the chance to explore more complicated animal movement models and make comparisons between these and the HMM.

Prerequisites: It is necessary for a prospective student to be able to code in R. Ideally, you will have taken MAS109 (formerly MAS113) and MAS2010 (formerly MAS223). Background knowledge on Bayesian statistics may be helpful but is by no means expected.

References: For background reading, it is suggested that you have a brief look at chapters 1-4 of the following paper:

- T. Patterson, A. Parton, R. Langrock, P. Blackwell, L. Thomas, & R. King (2017). Statistical modelling of individual animal movement: an overview of key methods and a discussion of practical challenges. AStA Advances in Statistical Analysis. 101(4), pp. 399-438. DOI: `https://doi.org/10.1007/s10182-017-0302-7`